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# Metric Regularity of the Sum of Multifunctions and Applications.

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**Abstract** In this work, we use the theory of error bounds to study metric regularity of the sum of two multifunctions, as well as some important properties of variational systems. We use an approach based on the metric regularity of epigraphical multifunctions. Our results subsume some recent results by Durea and Strugariu.

**Keywords** Error bound · Metric regularity · Pseudo-Lipschitz property · Sum-stability · Variational systems · Coderivative

**Mathematics Subject Classification (2000)** 49J52 · 49J53 · 49K27 · 90C34

## 1 Introduction

In this paper, we are especially interested in metric regularity of the sum of two multifunctions. The starting point of the study is the famous Lyusternik-Graves Theorem [1, 2], which reduces the problem of regularity of a strictly differentiable single-valued mapping between Banach spaces to that of its linear

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approximation. Historical comments and modern interpretations and extensions of this theorem can be found in [3, 4]. Particularly, it was observed in Dmitruk, Milyutin & Osmolovsky [4] that the original Lyusternik's proof in [1] is applicable to a much more general setting: the sum of a covering at a rate mapping and a Lipschitz one with suitable constants is covering at the rate. Extensions to the case of the sum of a metrically regular set-valued mapping and a single-valued Lipschitz map with suitable constants appear in [3, 5–10], (see the references therein for more details).

For the parametric case, it is well-known (see for instance, Dmitruk & Kruger [11], Aragón Artacho, Dontchev, Gaydu, Geoffroy, and Veliov [12]) that, if we perturb a metrically regular mapping $F$ by a mapping $g(\cdot,\cdot)$, Lipschitz with respect to $x$, uniformly in $p$, with a sufficiently small Lipschitz constant, then the perturbed mapping $F(\cdot)+g(\cdot,p)$ is metrically regular for every $p$ near $\bar{p}$. More generally, Ioffe [13] extended this result to the case of the sum of a metrically regular multifunction and a Lipschitz one, and also to the more general case, when if a multifunction $G$ is sufficiently close to the given metrically regular multifunction $F$ in the sense given in [13], then $G$ is necessarily metrically regular, with suitable constants (see also [14]).

When we perturb a metrically regular multifunction by another set-valued mapping which is pseudo-Lipschitz, the perturbed mapping, i.e., the sum set-valued mapping fails in general to be metrically regular, (we refer to the example in the next section). However, if for example the so-called " sum-stability" property (introduced below) holds, then the metric regularityf, as well as the pseudo-Lipschitz property of the variational system, remains. Recently, Durea & Strugariu [15] considered the sum of two set-valued mappings and obtained a result very similar to openness of the sum of two set-valued mappings. They also gave some applications to generalized variational systems.

Motivated by the ideas and results from [15], we attack these problems by using a different approach and with rather different assumptions. Indeed, using an approach based on the theory of error bounds, we study metric regularity of a special multifunction called the epigraphical multifunction associated to $F$ and $G$. This intermediate result allows us to study metric regularity/ linear openness of the sum of two set-valued mappings, as well as metric regularity of the general variational system, avoiding the strong assumption of the closedness of the sum multifunction.

The paper is structured as follows. Section 2 is devoted to preliminaries where we introduce the problem of generalized parametric inclusions. We give some illustrations through examples and we present a small survey on different notion of regularity. In Section 3, we recall some recent results on error bounds of parametrized systems and give, sometimes with some modifications, characterizations of metric regularity of multifunctions given in [14, 16]. In Section 4, in the context of Asplund spaces, we estimate the strong slope of the lower semicontinuous envelope of the distance function to the epigraphical mul-

tifunction associated to two given multifunctions $F$ and $G$. Then, we give sufficient conditions as well as a point-based condition for metric regularity of this epigraphical multifunction under a coderivative condition. In the last section, we study Robinson metric regularity and Aubin property of a generalized variational system.

## 2 Preliminaries

Generalized equations, i.e., inclusions of the type

$$0 \in F(x,p), \tag{1}$$

involving a multifunction $F : X \times P \rightrightarrows Y$ where $X, Y$ are metric spaces, and $P$ is a topological space considered as the space of parameters, have been extensively used for modeling optimization and complementarity problems, as well as variational inequalities since the pioneering work of Robinson [17,18]. The study of generalized equations constitute the core of the development of set-valued analysis [19] which is one of the main corner-stones of variational analysis, see, e.g., books [5,20–24]. A typical example of (1) is given by a parametrized system of inequalities/equalities. More precisely, let us consider the system $(\mathcal{S})$, consisting of those points $x$ for which

$$f_i(x,p) \le 0,\ i \in \{1, \cdots, k\},$$
$$f_i(x,p) = 0,\ i \in \{k+1, \cdots, k+d\},$$

where $x \in \mathbb{R}^m$ is the decision variable, $p \in \mathbb{R}^n$ a parameter and for each $i \in \{1, k+d\}$, and the $f_i's$ are functions from $\mathbb{R}^m \times \mathbb{R}^n$ to $\mathbb{R}$. Setting $f(x,p) = (f_1(x,p), \cdots, f_k(x,p), f_{k+1}(x,p), \cdots, f_{k+d}(x,p))$, and

$$F(x,p) := f(x,p) - \mathbb{R}^k_- \times \{0\}^d,$$

the system $(\mathcal{S})$ can be reformulated in the form (1). Let us also note that (1) includes the important subcase of parametrized *generalized inclusions*:

$$0 \in H(x) + f(x,p), \tag{2}$$

where $H : X \rightrightarrows Y$ is a set-valued mapping and $f : X \times P \to Y$ is a mapping.

Let us consider the perturbed optimization problem $(\mathcal{P})$

$$\min_{x \in C} [g(x) - \langle p, x \rangle],$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is a Fréchet differentiable function, and $p \in \mathbb{R}^n$ is a given parameter. The first order optimality condition of problem $(\mathcal{P})$ is given by

$$p - \nabla g(x) \in N_C(x). \tag{3}$$

Here $N_C$ stands for the normal cone mapping defined by

$$N_C(x) = \{v \in \mathbb{R}^n \,:\, \langle v, y - x\rangle \le 0 \quad \forall y \in C\}$$

if $x \in C$, and $N_C(x) = \emptyset$ otherwise. Setting $f(x,p) = \nabla g(x) - p$, relation (3) takes the form

$$0 \in f(x,p) + N_C(x). \tag{4}$$

Hence, the first order optimality condition satisfies the generalized variational inequality (4) and appears as a special case of equation (2).

The study of variational properties and stability of the solutions of equation (1) has attracted a large interest from a large number of authors, and we refer the reader to the monographs [5, 22, 24] and the references therein.

Let us first provide definitions and properties of some essential notions from set-valued analysis that will be used throughout this paper. In what follows, $X$, $Y$, etc., unless specified otherwise, are metric spaces, and we use the same symbol $d(\cdot,\cdot)$ to denote the distance in all of them or between a point $x$ to a subset $\Omega$ of one of them : $d(x,\Omega) := \inf_{u\in\Omega} d(x,u)$. By $B(x,\rho)$ and $\bar{B}(x,\rho)$ we denote the open and closed balls of radius $\rho$ around $x$, while, if $X$ is a normed linear space, we use the notations $B_X, \bar{B}_X$ for the open and the closed unit balls, respectively. By a multifunction (set-valued mapping) $S : X \rightrightarrows Y$, we mean a mapping from $X$ into the subsets (possibly empty) of $Y$. We denote by gph $S$ the graph of $S$, that is the set $\{(x,y) \in X \times Y \,:\, y \in S(x)\}$, and by $\mathrm{D}(S) := \{x \in X \,:\, S(x) \neq \emptyset\}$ the domain of $S$. When $S$ has a closed graph, we say that $S$ is a closed multifunction.

Since various types of multifunctions arise in a considerable number of models ranging from mathematical programs, through game theory and to control and design problems, they represent probably the most developed class of objects in variational analysis. A number of useful regularity properties have been introduced and investigated (see [5,24] and the references therein). Among them, the most popular is that of metric regularity ( [3, 5, 11, 13, 14, 19, 20, 22–35]), the root of which can be traced back to the classical Banach open mapping theorem and the subsequent fundamental results of Lyusternik and Graves ( [1,2]).

A multifunction $F$ is said to be *metrically regular* around $(\bar{x},\bar{y}) \in$ gph $F$ with modulus $\tau > 0$, whenever there exist neighborhoods $\mathcal{U},\mathcal{V}$ of $\bar{x},\bar{y}$, respectively, such that, for every $(x,y) \in \mathcal{U} \times \mathcal{V}$,

$$d(x, F^{-1}(y)) \le \tau d(y, F(x)). \tag{5}$$

A classical illustration of this concept concerns the case when $F$ is a bounded linear continuous operator. Then, metric regularity of $F$ amounts to saying that $F$ is surjective. In terms of the inverse mapping $S := F^{-1}$, property (5) can be rewritten equivalently as follows:

$$d(x, S(y)) \le \kappa d(y, y') \quad \forall y, y' \in \mathcal{V}, \forall x \in S(y') \cap \mathcal{U}. \tag{6}$$

This gives rise to another well known concept called *pseudo-Lipschitz property*, also called Lipschitz-like property (see [22]), or *Aubin property* (see [36]) at $(\bar{y}, \bar{x}) \in \operatorname{gph} S$ . The concept of *openness* or *covering (at a linear rate)* is also widely used: one says that $S : X \rightrightarrows Y$ is *open at linear rate* $\tau > 0$ around $(\bar{x}, \bar{y}) \in \operatorname{gph} S$ iff there exist neighborhoods $\mathcal{U}, \mathcal{V}$ of $\bar{x}, \bar{y}$, respectively and, a positive number $\varepsilon > 0$ such that, for every $(x, y) \in \operatorname{gph} S \cap (\mathcal{U} \times \mathcal{V})$ and every $\rho \in ]0, \varepsilon[$,

$$B(y, \rho\tau) \subset S(B(x, \rho)).$$

We refer to [3,4,22,24,25,27,36,37] and the references therein for different developments of these notions. The following relation is well established:

$$\text{Metric regularity} \iff \text{Covering} \iff \text{Aubin property of the inverse.} \tag{7}$$

Let us also add that in Banach spaces, similarly to the classical calculus, one can formulate sufficient (sub)differential characterizations of properties (5) and (6) (see, e.g., [3,22,37]). In Asplund spaces (see [22, 38] for definitions and characterizations of Asplund spaces), the corresponding characterizations in terms of Fréchet subdifferentials ( [39, 40]) or their limiting counterparts ( [22, 41–43]) and the corresponding coderivatives become necessary and sufficient.

From the point of view of applications to optimization (sensitivity analysis, convergence analysis of algorithms, and penalty functions methods), one of the most important regularity properties seems to be that of error bounds, providing an estimate for the distance of a point from the solution set. This theory was initiated by the pioneering work by Hoffman [44][1]. A general classification scheme of necessary and sufficient criteria for the error bound property is presented in [46, 47]. Applications of the theory of error bounds to the investigation of metric regularity of multifunctions have been recently studied and developed by many authors, including for instance [8, 14, 16, 48–51].

## 3 Metric Regularity of Epigraphical Multifunctions via Error Bounds

Let us remind some basic notions used in the paper. Let $f : X \to \mathbb{R} \cup \{+\infty\}$ be a given extended-real-valued function. As usual, $\operatorname{Dom} f := \{x \in X : f(x) < +\infty\}$ denotes the domain of $f$. We recall the concept of error bounds that is one of the most important regularity properties. We set

$$S := \{x \in X : \quad f(x) \leq 0\}, \tag{8}$$

and we use the symbol $[f(x)]_+$ to denote $\max\{f(x), 0\}$. We say that $f$ satisfies the *an error bound property* iff there exists a real $c > 0$ such that

$$d(x, S) \leq c\big[f(x)\big]_+ \quad \text{for all} \quad x \in X. \tag{9}$$

[1] It has been pointed out recently to the authors by Hiriart-Urruty that traces of the error bound property were already in [45], published in 1951.

For $x_0 \in S$, $f$ has a local error bound at $x_0$, when there exist reals $c > 0$ and $\varepsilon > 0$ such that (9) is satisfied for all $x$ around $x_0$, i.e., in an open ball $B(x_0, \varepsilon)$.

Given a multifunction $F : X \rightrightarrows Y$, we make use of the lower semicontinuous envelope $(x, y) \mapsto \varphi_F(x, y)$ of the function $(x, y) \mapsto d(y, F(x))$, i.e., for $(x, y) \in X \times Y$,

$$\varphi_F(x, y) := \liminf_{(u,v)\to(x,y)} d(v, F(u)) = \liminf_{u\to x} d(y, F(u)). \tag{10}$$

Recall from De Giorgi, Marino & Tosques [52], that the strong slope $|\nabla f|(x)$ of a lower semicontinuous function $f$ at $x \in \text{Dom } f$ is the quantity defined by $|\nabla f|(x) = 0$ if $x$ is a local minimum of $f$, and f

$$|\nabla f|(x) = \limsup_{y\to x, y\neq x} \frac{f(x) - f(y)}{d(x, y)},$$

otherwise. For $x \notin \text{Dom } f$, we set $|\nabla f|(x) = +\infty$.

We now consider a parametrized inequality system, that is, the problem of finding $x \in X$ such that

$$f(x, p) \leq 0, \tag{11}$$

where $f : X \times P \to \mathbb{R} \cup \{+\infty\}$ is an extended-real-valued function, $X$ is a complete metric space and $P$ is a topological space. We denote by $S(p)$ the set of solutions of system (11):

$$S(p) := \{x \in X : \quad f(x, p) \leq 0\}.$$

The following theorem ( [16, Theorem 2]) gives necessary and sufficient conditions for the existence of a local uniform error bound for the parametric system (11).

**Theorem 3.1** *Let $X$ be a complete metric space and $P$ be a topological space. Suppose that the mapping $f : X \times P \to \mathbb{R} \cup \{+\infty\}$ satisfies the following conditions for some $(\bar{x}, \bar{p}) \in X \times P$ :*

*(a) $\bar{x} \in S(\bar{p})$;*

*(b) the mapping $p \mapsto f(\bar{x}, p)$ is upper semicontinuous at $\bar{p}$;*

*(c) for any $p$ near $\bar{p}$, the mapping $x \mapsto f(x, p)$ is lower semicontinuous near $\bar{x}$.*

*Let $\tau > 0$ be given. The the following two statements are equivalent:*

*(i) There exists a neighborhood $\mathcal{V} \times \mathcal{W} \subseteq X \times P$ of $(\bar{x}, \bar{p})$ such that for any $p \in \mathcal{W}$, we have $\mathcal{V} \cap S(p) \neq \emptyset$ and*

$$d(x, S(p)) \leq \tau[f(x, p)]_+ \quad \textit{for all} \quad (x, p) \in V \times \mathcal{W}. \tag{12}$$

*(ii) There exist a neighborhood $\mathcal{V} \times \mathcal{W} \subseteq X \times P$ of $(\bar{x}, \bar{p})$ and a real $\gamma > 0$ such that for each $(x, p) \in \mathcal{V} \times \mathcal{W}$ with $f(x, p) \in (0, \gamma)$ and for any $\varepsilon > 0$, we can find $z \in X$ such that*

$$0 < d(x, z) < (\tau + \varepsilon)(f(x, p) - [f(z, p)]_+). \tag{13}$$

Given metric spaces $X, Y$ and a topological space $P$, we next consider the implicit multifunction : $X \times P \rightrightarrows Y$ defined by

$$S(y,p) := \{x \in X : \quad y \in F(x,p)\}. \tag{14}$$

Similarly to (10), we use the lower semicontinuous envelope $(x,y) \mapsto \varphi_p(x,y)$ of the function $(x,y) \mapsto d(y, F(x,p))$ for each $p \in P$, i.e., for $(x,y) \in X \times Y$,

$$\varphi_p(x,y) := \liminf_{(u,v)\to(x,y)} d(v, F(u,p)) = \liminf_{u\to x} d(y, F(u,p)). \tag{15}$$

From now on, we will also use the notation $F_p$ for $F(\cdot, p)$ and $\varphi_p$ for $\varphi_{F_p}$ and the metric defined on the cartesian product $X \times Y$ is given by:

$$d((x,y),(u,v)) = \max\{d(x,u), d(y,v)\}, \quad (x,y),(u,v) \in X \times Y.$$

The next lemma is useful.

**Lemma 3.1** *We suppose that the set-valued mapping $x \rightrightarrows F(x,p)$ is a closed multifunction (i.e., its graph is closed) for any $p$ near $\bar{p}$. Then, for each $y \in Y$, and each $p$ near $\bar{p}$,*

$$S(y,p) = \{x \in X : \ \varphi_p(x,y) = 0\}.$$

**Theorem 3.2** *Let $X$ be a complete metric space and $Y$ be a metric space. Let $P$ be a topological space and suppose that the set-valued mapping $F : X \times P \rightrightarrows Y$ satisfies the following conditions for some $(\bar{x}, \bar{y}, \bar{p}) \in X \times Y \times P$ :*

*(a) $\bar{x} \in S(\bar{y}, \bar{p})$;*

*(b) the multifunction $p \rightrightarrows F(\bar{x}, p)$ is lower semicontinuous at $\bar{p}$;*

*(c) for any $p$ near $\bar{p}$, the set-valued mapping $x \rightrightarrows F(x,p)$ is a closed multifunction (i.e., its graph is closed).*

*Let $\tau \in (0, +\infty)$, be fixed. Then one has the following implications: (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iii) $\Leftarrow$ (iv). Moreover, all the assertions are equivalent provided that $Y$ is a normed space.*

*(i) There exists a neighborhood $\mathcal{U} \times \mathcal{V} \times \mathcal{W} \subseteq X \times P \times Y$ of $(\bar{x}, \bar{y}, \bar{p})$ such that $\mathcal{V} \cap S(y,p) \neq \emptyset$ for any $(y,p) \in \mathcal{V} \times \mathcal{W}$ and*

$$d(x, S(y,p)) \leq \tau d(y, F(x,p)) \quad \textit{for all } (x,y,p) \in \mathcal{U} \times \mathcal{V} \times \mathcal{W};$$

*(ii) There exists a neighborhood $\mathcal{U} \times \mathcal{V} \times \mathcal{W} \subseteq X \times P \times Y$ of $(\bar{x}, \bar{y}, \bar{p})$ such that $\mathcal{V} \cap S(y,p) \neq \emptyset$ for any $(y,p) \in \mathcal{V} \times \mathcal{W}$ and*

$$d(x, S(y,p)) \leq \tau \varphi_p(x,y) \quad \textit{for all } (x,y,p) \in \mathcal{U} \times \mathcal{V} \times \mathcal{W};$$

*(iii) There exist a neighborhood $\mathcal{U}\times\mathcal{V}\times\mathcal{W}\subseteq X\times Y\times P$ of $(\bar{x},\bar{y},\bar{p})$ and a real $\gamma\in(0,+\infty)$ such that for any $(x,y,p)\in\mathcal{U}\times\mathcal{V}\times\mathcal{W}$ with $y\notin F(x,p)$ and any $\varepsilon>0$, and any sequence $\{x_n\}_{n\in\mathbb{N}}\subseteq X$ converging to $x$ with*

$$\lim_{n\to\infty} d(y,F(x_n,p)) = \liminf_{u\to x} d(y,F(u,p)) = \varphi_p(x,p),$$

*there exists a sequence $\{u_n\}_{n\in\mathbb{N}}\subseteq X$ with $\liminf_{n\to\infty} d(u_n,x)>0$ such that*

$$\limsup_{n\to\infty}\frac{d(y,F(x_n,p))-d(y,F(u_n,p))}{d(x_n,u_n)} > \frac{1}{\tau+\varepsilon}; \tag{16}$$

*(iv) There exist a neighborhood $\mathcal{U}\times\mathcal{V}\times\mathcal{W}\subseteq X\times P\times Y$ of $(\bar{x},\bar{p},\bar{y})$ and a real $\gamma>0$ such that*

$$|\nabla\varphi_p(\cdot,y)|(x)\geq\frac{1}{\tau}\quad \textit{for all } (x,y,p)\in\mathcal{U}\times\mathcal{V}\times\mathcal{W}\quad \textit{with } \varphi_p(x,y)\in(0,\gamma). \tag{17}$$

*Proof.* The implications $(ii)\Rightarrow(i)$ and $(iv)\Rightarrow(iii)$ are obvious. For $(i)\Rightarrow(iii)$, let $\mathcal{U}\times\mathcal{V}\times\mathcal{W}$ be an open neighborhood of $(\bar{x},\bar{y},\bar{p})$ such that gph $F(\cdot,p)$ is closed for $p\in\mathcal{W}$ and

$$d(x,S(y,p))\leq\tau d(y,F(x,p))\ \ \forall(x,y,p)\in\mathcal{U}\times\mathcal{V}\times\mathcal{W}.$$

Let $(x,y,p)\in\mathcal{U}\times\mathcal{V}\times\mathcal{W},\ y\notin F(x,p)$ and $\varepsilon>0$. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence converging to $x$. When $n$ is sufficiently large, say $n\geq n_0$, then $x_n\in\mathcal{U}$ as well as $y\notin F(x_n,p)$. Hence $d(x_n,S(y,p))\leq\tau d(y,F(x_n,p))$. For each $n\geq n_0$, pick $u_n\in S(y,p)$ such that $d(x_n,u_n)<(1+\varepsilon/2\tau)d(x_n,S(y,p))$. We claim that $\liminf_{n\to\infty} d(u_n,x)>0$. Otherwise, there would exist some subsequence, $\{u_{n_k}\}_{k\in\mathbb{N}}$ converging to $x$ such that $u_{n_k}\in S(y,p), i.e., y\in F(u_{n_k},p)$. Then, since $F(\cdot,p)$ is graph-closed this would imply $y\in F(x,p)$, a contradiction. Moreover for all $n\geq n_0$,

$$d(x_n,u_n)<(1+\varepsilon/2\tau)d(x_n,S(y,p))\leq(\tau+\varepsilon/2)[d(y,F(x_n,p))-d(y,F(u_n,p))].$$

This shows that (20) holds.

For $(iii)\Rightarrow(ii)$. Since the multifunction $p\rightrightarrows F(\bar{x},p)$ is assumed to be lower semicontinuous at $\bar{p}$, then the function $(p,y)\mapsto d(y,F(\bar{x},p))$ is upper semicontinuous at $(\bar{p},\bar{y})$ (see, e.g., in [53, Cor. 20 ]). Therefore,

$$\limsup_{(p,y)\to(\bar{p},\bar{y})}\varphi_p(\bar{x},y)\leq\limsup_{(p,y)\to(\bar{p},\bar{y})} d(y,F(\bar{x},p))\leq d(\bar{y},F(\bar{x},\bar{p}))=\varphi_{\bar{p}}(\bar{x},\bar{y}).$$

That is, the function $(p,y)\mapsto\varphi_p(\bar{x},y)$ is upper semicontinuous at $(\bar{p},\bar{y})$, and therefore, by virtue of Theorem 3.1, it suffices to observe that statement (ii) of Theorem 3.1 is verified. Indeed, let $(x,y,p)\in\mathcal{U}\times\mathcal{V}\times\mathcal{W}$ with $y\notin F(x,p)$ and $\varphi_p(x,y)<\gamma$ and let $\varepsilon\in(0,1)$ be given. Let $\{x_n\}_{n\in\mathbb{N}}$ be a sequence converging to $x$ with

$$\lim_{n\to\infty} d(y,F(x_n,p))=\varphi_p(x,y)=\liminf_{u\to x} d(y,F(u,p)).$$

Then, $x_n \notin F_p^{-1}(y)$, i.e., $y \notin F(x_n, p)$ when $n$ is sufficiently large, say $n \geq n_0$. By (iii), we consider a sequence $\{u_n\}_{n\in\mathbb{N}}$ with $\liminf_{n\to\infty} d(u_n, x) > 0$ such that

$$\limsup_{n\to\infty} \frac{d(y, F(x_n, p)) - d(y, F(u_n, p))}{d(x_n, u_n)} > \frac{1}{\tau + \varepsilon}.$$

Pick $\delta \in (0, \liminf_{n\to\infty} d(u_n, x))$. Then, take an index $n_1 \geq n_0$ such that for all $n \geq n_1$, we have

$$d(x_n, u_n) \geq \delta; d(x_n, x) < \varepsilon\delta; d(y, F(x_n, p) < \varphi_p(x, y) + \frac{\varepsilon}{\tau + \varepsilon} d(x_n, u_n)$$

and

$$d(x_n, u_n) < (\tau + \varepsilon)(d(y, F(x_n, p)) - d(y, F(u_n, p))).$$

Hence,

$$d(x_n, u_n) < (1 - \varepsilon)^{-1}(\tau + \varepsilon)(\varphi_p(x, y) - \varphi_p(u_n, y)).$$

It follows that for all $n \geq n_1$,

$$\begin{aligned} d(x, u_n) &\leq (1 + \varepsilon) d(x_n, u_n) \\ &< (1 - \varepsilon)^{-1}(\tau + \varepsilon)(1 + \varepsilon)(\varphi_p(x, y) - \varphi_p(u_n, y)) \\ &< (\tau + \varepsilon)(\varphi_p(x, y) - \varphi_p(u_n, y)) \end{aligned}$$

and statement (ii) of Theorem 3.1 follows directly. So, the implication $(iii) \Rightarrow (ii)$ is now proved.

When $Y$ is normed space, $(i) \Rightarrow (iv)$ follows from the converse part of [16, Theorem 5]) by noting that $S(y, p) = F_p^{-1}(y)$. So, we have that all assertions are equivalent when $Y$ to be normed space. The proof is complete. △

Given two multifunctions $F, G : X \rightrightarrows Y$, ($Y$ is a normed linear space) we define a new multifunction $\mathcal{E}_{(F,G)} : X \times Y \rightrightarrows Y$ by setting

$$\mathcal{E}_{(F,G)}(x, k) = \begin{cases} F(x) + k, \text{ if } k \in G(x), \\ \emptyset, \qquad\qquad \text{otherwise.} \end{cases}$$

When one of the multifunctions is a cone, $\mathcal{E}_{(F,G)}$was called *epigraphical* by Durea and Strugariu [15].

For given $y \in Y$, we set

$$\mathbb{S}_{\mathcal{E}_{(F,G)}}(y) := \{(x, k) \in X \times Y : \ \ y \in \mathcal{E}_{(F,G)}(x, k)\}. \tag{18}$$

The lower semicontinuous envelope $((x, k), y) \mapsto \varphi_{\mathcal{E}}((x, k), y)$ of the distance function $d(y, \mathcal{E}_{(F,G)}(x, k))$ is defined for $(x, k, y) \in X \times Y \times Y$ by

$$\varphi_{\mathcal{E}}((x, k), y) := \liminf_{(u,v,w)\to(x,k,y)} d(w, \mathcal{E}_{(F,G)}(u, v)).$$

Let us recall that a multifunction $G : X \rightrightarrows Y$ is lower semicontinuous at $(x, y) \in \text{gph } G$, if for any sequence $\{x_n\}_{n\in N}$ converging to $x$, we can provide a sequence $\{y_n\}_{n\in\mathbb{N}}$ converging to $y$, with $y_n \in G(x_n)$.

**Lemma 3.2** *If $G$ has closed graph then*

$$\varphi_{\mathcal{E}}((x,k),y) = \begin{cases} \liminf\limits_{\text{gph } G \ni (u,v) \to (x,k)} d(y, F(u)+v), & \text{if } k \in G(x) \\ +\infty, & \text{otherwise.} \end{cases}$$

*Moreover, if in addition, $G$ is lower semicontinuous at $(x,k) \in \text{gph } G$, then the following representation holds:*

$$\varphi_{\mathcal{E}}((x,k),y) = \begin{cases} \liminf\limits_{u \to x} d(y, F(u)+k), & \text{if } k \in G(x) \\ +\infty, & \text{otherwise.} \end{cases}$$

*Proof* For the first equality, if $k \notin G(x)$, since $G$ has closed graph one has $\varphi_{\mathcal{E}}((x,k),y) = \infty$. Otherwise, we have

$$\begin{aligned} \varphi_{\mathcal{E}}((x,k),y = &\liminf_{(u,v,w)\to(x,k,y)} d(w, \mathcal{E}_{(F,G)}(u,v)) \\ = &\liminf_{\text{gph } G \ni (u,v) \to (x,k)} d(y, F(u)+v). \end{aligned}$$

**Claim** Let $G : X \rightrightarrows Y$ be lower semicontinuoous at $(x,k) \in \text{gph } G$. Then for each $y \in Y$ we have

$$\liminf_{\text{gph } G \ni (u,v) \to (x,k)} d(y, F(u)+v) = \liminf_{u \to x} d(y, F(u)+k).$$

For simplicity set $A := \liminf_{\text{gph } G \ni (u,v) \to (x,k)} d(y, F(u)+v)$ and $B := \liminf_{u\to x} d(y, F(u)+k)$. First let us prove that $A \geq B$. Indeed, let $\{(u_n, v_n)\}_{n\in\mathbb{N}}$ be a sequence in gph $G$ such that $(u_n, v_n) \to (x,k)$ as $n \to +\infty$ and $\lim_{n\to+\infty} d(y, F(u_n)+v_n) = A$. Then,

$$\begin{aligned} B \leq \liminf_{n\to+\infty} d(y, F(u_n)+k) &\leq \liminf_{n\to+\infty} [d(y, F(u_n)+v_n) + \|v_n - k\|] \\ &= \lim_{n\to+\infty} d(y, F(u_n)+v_n) = A. \end{aligned}$$

On the other hand, to prove that $A \leq B$, pick any sequence $\{u_n\}_{n\in\mathbb{N}}$ converging to $x$ such that $\lim_{n\to+\infty} d(y, F(u_n)+k) = B$. As $G$ is lower semicontinuous at $(x,k)$, we find a sequence $\{v_n\}_{n\in\mathbb{N}}$ converging to $k$ such that $(u_n, v_n) \in \text{gph } G$ for each $n \in \mathbb{N}$. Hence,

$$\begin{aligned} A &\leq \liminf_{n\to+\infty} d(y, F(u_n)+v_n) \\ &\leq \liminf_{n\to+\infty} [d(y, F(u_n)+k + \|k - v_n\|] \\ &\leq \lim_{n\to+\infty} d(y, F(u_n)+k) = B. \end{aligned}$$

The claim is proved. From the claim, the fact that $\varphi_{\mathcal{E}}((x,k),y) = \liminf_{u\to x} d(y, F(u)+k)$ follows immediately. △

*Remark 3.1* (i) Since we suppose that $G$ is both graph-closed and lower semicontinuous, it is continuous in finite dimension (see, [24, Theorem 5.7 page 158]).

(ii) The lower semicontinuity of $G$ is necessary to obtain the last formula in Lemma 3.2 as shows the next example[2]: take $F, G : [0,1] \rightrightarrows \mathbb{R}$ be defined by $F(0) = \{0\}, F(x) = 1$ if $x \in (0,1]$ and $G(0) = \{0,1\}, G(x) = \{0\},$ if $x \in (0,1]$. Note that $G$ has a closed graph but is not lower semicontinuous at $(0,1) \in \operatorname{gph} G$ and remark that

$$\liminf_{u\to 0} d(3, F(u)+1) = 1 \text{ while } \varphi_{\mathcal{E}}((0,3),1) = \liminf_{(u,v,w)\to(0,1,3)} d(w, \mathcal{E}_{(F,G)}(u,v)) = 2.$$

The next lemma is useful.

**Lemma 3.3** *Assume that $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Y$ be closed multifunctions. Then, the epigraphical multifunction $\mathcal{E}_{(F,G)}$ has a closed graph, and for each $y \in Y$,*

$$\mathbb{S}_{\mathcal{E}_{(F,G)}}(y) = \{(x,k) \in X \times Y : \quad \varphi_{\mathcal{E}}((x,k),y) = 0\} = \{(x,k) \in X \times Y : \quad k \in G(x), y \in F(x) + k\}. \tag{19}$$

*Proof* Observe that, if $F : X \rightrightarrows Y$ and $G : X \rightrightarrows Y$ are closed multifunctions, then so is the epigraphical multifunction $\mathcal{E}_{(F,G)}$.

Let us prove (19). Obviously, for each $y \in Y$, if $(x,k) \in \mathbb{S}_{\mathcal{E}_{(F,G)}}(y)$, then $\varphi_{\mathcal{E}}((x,k),y) = 0$. Conversely, suppose that $\varphi_{\mathcal{E}}((x,k),y) = 0$. Then, $k \in G(x)$ and there exists a sequence $\{(x_n,k_n)\} \to (x,k), k_n \in G(x_n)$ such that $d(y, F(x_n)+k_n) \to 0$. Then, there exists $z_n \in F(x_n)$ such that $z_n + k_n \to y$. It follows, $z_n \to y-k$. Since $F$ is graph-closed, one has that $y-k \in F(x)$, i.e., $y \in F(x)+k$. Hence, $(x,k) \in \mathbb{S}_{\mathcal{E}_{(F,G)}}(y)$ establishing the proof. △

By virtue of Lemma 3.3, we adapt Theorem 3.2 to the multifunction $\mathcal{E}_{(F,G)}$.

**Lemma 3.4** *Let $X$ be a complete metric space, let $Y$ be a Banach space and let $F, G : X \rightrightarrows Y$ be closed multifunctions. Suppose that $(\bar{x}, \bar{k}, \bar{y}) \in X \times Y \times Y$ such that $\bar{y} \in F(\bar{x}) + \bar{k}, \bar{k} \in G(\bar{x})$.*

*Let $\tau \in ]0,+\infty[$, be fixed. Then, the following statements are equivalent:*

*(i) There exists a neighborhood $\mathcal{U} \times \mathcal{V} \times \mathcal{W} \subseteq X \times Y \times Y$ of $(\bar{x}, \bar{k}, \bar{y})$ such that $(\mathcal{U} \times \mathcal{V}) \cap \mathbb{S}_{\mathcal{E}_{(F,G)}}(y) \neq \emptyset$ for any $y \in \mathcal{W}$ and*

$$d((x,k), \mathbb{S}_{\mathcal{E}_{(F,G)}}(y)) \leq \tau \varphi_{\mathcal{E}}((x,k),y) \quad \textit{for all} \quad (x,k,y) \in \mathcal{U} \times \mathcal{V} \times \mathcal{W};$$

*(ii) There exist a neighborhood $\mathcal{U} \times \mathcal{V} \times \mathcal{W} \subseteq X \times Y \times Y$ of $(\bar{x}, \bar{k}, \bar{y})$ and a real $\gamma \in ]0,+\infty[$ such that, for any $(x,k,y) \in \mathcal{U} \times \mathcal{V} \times \mathcal{W}$ with $y \notin F(x)+k, k \in G(x)$ and $\varphi_{\mathcal{E}}((x,k),y) < \gamma$, any $\varepsilon > 0$, and any sequences $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converging to $x$, $\{k_n\}_{n\in\mathbb{N}} \subseteq Y$ converging to $k$, $k_n \in G(x_n)$ with*

$$\lim_{n\to\infty} d(y-k_n, F(x_n)) = \liminf_{\operatorname{gph} G \ni (u,v) \to (x,k)} d(y-v, F(u)),$$

[2] We would like to thank one of the referees for pointing us this example.

*there exist sequences* $\{u_n\}_{n\in\mathbb{N}} \subseteq X,\ \{z_n\}_{n\in\mathbb{N}} \subseteq Y$ *with* $(u_n, z_n) \in \operatorname{gph} G$ *and* $\liminf_{n\to\infty} d((u_n, z_n), (x, k)) > 0$ *such that*

$$\limsup_{n\to\infty} \frac{d(y - k_n, F(x_n)) - d(y - z_n, F(u_n))}{d((x_n, u_n), (k_n, z_n))} > \frac{1}{\tau + \varepsilon}; \tag{20}$$

*(iii) there exist a neighborhood* $\mathcal{U} \times \mathcal{V} \times \mathcal{W}$ *of* $(\bar{x}, \bar{k}, \bar{y})$ *and a real* $\gamma > 0$ *such that*

$$|\nabla \varphi_{\mathcal{E}}((\cdot, \cdot), y)|(x, k) \geq \frac{1}{\tau} \text{ for all } (x, k, y) \in \mathcal{U} \times \mathcal{V} \times \mathcal{W} \text{ with } \varphi_{\mathcal{E}}((x, k), y) \in ]0, \gamma[.$$

**Proposition 3.1** *Let* $X$ *be a complete metric space,* $Y$ *be a Banach space and let* $F, G : X \rightrightarrows Y$ *be closed multifunctions. Suppose that* $(\bar{x}, \bar{k}, \bar{y}) \in X \times Y \times Y$ *be such that* $\bar{y} \in F(\bar{x}) + \bar{k}, \bar{k} \in G(\bar{x})$*. Consider the following statements:*

*(i) there exist a neighborhood* $\mathcal{U} \times \mathcal{V} \times \mathcal{W}$ *of* $(\bar{x}, \bar{k}, \bar{y})$ *and* $\tau > 0$ *such that*

$$d((x, k), \mathbb{S}_{\mathcal{E}_{(F,G)}}(y)) \leq \tau \varphi_{\mathcal{E}}((x, k), y) \quad \textit{for all} \quad (x, k, y) \in \mathcal{U} \times \mathcal{V} \times \mathcal{W};$$

*(ii) there exist a neighborhood* $\mathcal{U} \times \mathcal{V} \times \mathcal{W}$ *of* $(\bar{x}, \bar{k}, \bar{y})$ *and* $\tau > 0$ *such that*

$$d(x, (F + G)^{-1}(y)) \leq \tau d(y, F(x) + G(x) \cap \mathcal{V}) \quad \textit{for all} \quad (x, y) \in \mathcal{U} \times \mathcal{W}; \tag{21}$$

*(iii) there exist a neighborhood* $\mathcal{U} \times \mathcal{V} \times \mathcal{W}$ *of* $(\bar{x}, \bar{k}, \bar{y} - \bar{k})$ *and* $\varepsilon, \tau > 0$ *such that, for every* $(x, k, z) \in \mathcal{U} \times \mathcal{V} \times \mathcal{W}, k \in G(x), z \in F(x),$ *and* $\rho \in ]0, \varepsilon[$,

$$B(k + z, \rho\tau^{-1}) \subset (F + G)(B(x, \rho)).$$

*Then one has the following implications:* $(i) \Rightarrow (ii) \Leftrightarrow (iii)$.

*Proof* For $(i) \Rightarrow (ii)$. By (i), there exist $\delta_1, \delta_2, \delta_3 > 0$ such that, for every $\varepsilon > 0$ and for every $(x, k, y) \in B(\bar{x}, \delta_1) \times [B(\bar{k}, \delta_2) \cap G(x)] \times B(\bar{y}, \delta_3)$, there is $(u, z) \in X \times Y$ with $y \in F(u) + z, z \in G(u)$ such that

$$d((x, k), (u, z)) < (1 + \varepsilon)\tau \varphi_{\mathcal{E}}((x, k), y).$$

Consequently,

$$d(x, u) \leq \max\{d(x, u), \|k - z\|\} < (1 + \varepsilon)\tau d(y, F(x) + k).$$

Noting that $y \in F(u) + G(u)$, i.e., $u \in (F + G)^{-1}(y)$, it follows that

$$d(x, (F + G)^{-1}(y)) < (1 + \varepsilon)\tau d(y, F(x) + k).$$

In conclusion, we have that

$$d(x, (F + G)^{-1}(y)) < (1 + \varepsilon)\tau d(y, F(x) + G(x) \cap B(\bar{k}, \delta_2)) \quad \text{for all} \quad (x, y) \in B(\bar{x}, \delta_1) \times B(\bar{y}, \delta_3).$$

Hence, taking the limit as $\varepsilon > 0$ goes to 0 yields the desired conclusion.

For $(ii) \Rightarrow (iii)$. Suppose that (ii) holds for the neighborhood $B(\bar{x}, \delta_1) \times B(\bar{k}, \delta_2) \times B(\bar{y}, \delta_3)$ with $\delta_1, \delta_2, \delta_3 > 0$ and $\tau > 0$. Choose $\rho_1 = \delta_1, \rho_2 = 1/4 \min\{\delta_2, \delta_3\}, \rho_3 = 1/4\delta_3, \varepsilon < \tau\delta_3/2$.

Then, for $(x,k,z) \in B(\bar{x},\rho_1) \times B(\bar{k},\rho_2) \times B(\bar{y}-\bar{k},\rho_3), k \in G(x), z \in F(x)$, we take $y \in B(k+z,\rho\tau^{-1})$.

Consequently,

$$\|y-k-z\| < \rho\tau^{-1},$$

and

$$\begin{aligned}\|y-\bar{y}\| &\leq \|y-k-z\| + \|k-\bar{k}\| + \|\bar{k}-\bar{y}+z\|,\\ &< \rho\tau^{-1} + \rho_2 + \rho_3,\\ &< \varepsilon\tau^{-1} + \delta_3/4 + \delta_3/4,\\ &< \delta_3/2 + \delta_3/2 = \delta_3.\end{aligned}$$

Therefore, we have that

$$d(y, F(x) + G(x) \cap B(\bar{k},\delta_2)) \leq \|y-k-z\| < \rho\tau^{-1}.$$

Hence,

$$d(x,(F+G)^{-1}(y)) < \tau\rho\tau^{-1} = \rho.$$

Let $\gamma > 0$ with $d(x,(F+G)^{-1}(y)) + \gamma < \rho$. Find $u \in (F+G)^{-1}(y)$, i.e., $y \in (F+G)(u)$ such that

$$d(x,u) < d(x,(F+G)^{-1}(y)) + \gamma.$$

Thus, $d(x,u) < \rho$. It follows that

$$y \in (F+G)(B(x,\rho)).$$

For $(iii) \Rightarrow (ii)$. Suppose that (iii) holds for the neighborhood $B(\bar{x},\rho_1) \times B(\bar{k},\rho_2) \times B(\bar{y},\rho_3)$ with $\rho_1,\rho_2,\rho_3 > 0$ and $\tau > 0, \varepsilon > 0$.

Take $\rho_1,\rho_3$ smaller if neccesary and consider a positive real $\eta$ sufficiently small so that the quantity $\rho := \tau d(y, F(x) + G(x) \cap B(\bar{k},\rho_2)) + \eta$ satisfies the conclusion of (iii) together with $y \in B(k+z,\rho\tau^{-1})$. Then, there is a $u \in B(x,\rho)$ such that $y \in (F+G)(u)$, that is, $u \in (F+G)^{-1}(y)$.

Thus,

$$d(x,(F+G)^{-1}(y)) \leq d(x,u) < \rho = \tau d(y, F(x) + G(x) \cap B(\bar{k},\rho_2)) + \eta.$$

Since $\eta > 0$ is arbitrary, the proof is complete. △

The next result gives conditions for the sum of two metrically regular mappings $F, G$ to remain metrically regular. Before stating this result, we need to recall the so-called "locally sum-stable" property introduced in [15].

**Definition 3.1** Let $F, G : X \rightrightarrows Y$ be two multifunctions and $(\bar{x},\bar{y},\bar{z}) \in X \times Y \times Y$ such that $\bar{y} \in F(\bar{x}), \bar{z} \in G(\bar{x})$. We say that the pair $(F,G)$ is locally sum-stable around $(\bar{x},\bar{y},\bar{z})$ iff for every $\varepsilon > 0$, there exists $\delta > 0$ such that, for every $x \in B(\bar{x},\delta)$ and every $w \in (F+G)(x) \cap B(\bar{y}+\bar{z},\delta)$, there are $y \in F(x) \cap B(\bar{y},\varepsilon)$ and $z \in G(x) \cap B(\bar{z},\varepsilon)$ such that $w = y + z$.

A simple case which ensures the local sum-stability of $(F, G)$ is as follows.

**Proposition 3.2** *Let $F : X \rightrightarrows Y, G : X \rightrightarrows Y$ be two multifunctions and $(\bar{x}, \bar{y}, \bar{z}) \in X \times Y \times Y$ such that $\bar{y} \in F(\bar{x}), \bar{z} \in G(\bar{x})$. If $G(\bar{x}) = \{\bar{z}\}$ and $G$ is upper semicontinuous at $\bar{x}$, then the pair $(F, G)$ is locally sum-stable around $(\bar{x}, \bar{y}, \bar{z})$.*

*Proof* Since $G$ is upper semicontinuous at $\bar{x}$, for every $\varepsilon > 0$ there exists $\delta > 0$ such that

$$G(x) \subset G(\bar{x}) + B(0, \varepsilon/2) = \bar{z} + B(0, \varepsilon/2) = B(\bar{z}, \varepsilon/2), \quad \text{for all } x \in B(\bar{x}, \delta).$$

Set

$$\eta := \min\{\delta, \varepsilon/2\}$$

and take $x \in B(\bar{x}, \eta)$ and $w \in (F + G)(x) \cap B(\bar{y} + \bar{z}, \eta)$. Then, there are $y \in F(x), z \in G(x)$ such that

$$w = y + z \text{ and } w \in B(\bar{y} + \bar{z}, \eta).$$

Clearly, $z \in B(\bar{z}, \varepsilon/2) \subset B(\bar{z}, \varepsilon)$.

Moreover,

$$\|y - \bar{y}\| = \|w - z - \bar{y}\| \le \|w - \bar{y} - \bar{z}\| + \|z - \bar{z}\| < \eta + \varepsilon/2 \le \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Consequently,

$$w = y + z, y \in F(x) \cap B(\bar{y}, \varepsilon), z \in G(x) \cap B(\bar{z}, \varepsilon).$$

Hence we have established that $(F, G)$ is locally sum-stable around $(\bar{x}, \bar{y}, \bar{z})$. △

**Proposition 3.3** *Let $X$ be a complete metric space, $Y$ be a Banach space and let $F, G : X \rightrightarrows Y$ be closed multifunctions. Suppose that $(\bar{x}, \bar{k}, \bar{y}) \in X \times Y \times Y$ is such that $\bar{y} \in F(\bar{x}) + \bar{k}, \bar{k} \in G(\bar{x})$.*
*If the pair $(F, G)$ is locally sum-stable around $(\bar{x}, \bar{y} - \bar{k}, \bar{k})$ and there exist a neighborhood $\mathcal{U} \times \mathcal{V}$ of $(\bar{x}, \bar{y})$ and $\tau, \theta > 0$ such that*

$$d(x, (F + G)^{-1}(y)) \le \tau d(y, F(x) + G(x) \cap B(\bar{k}, \theta)) \quad \text{for all} \quad (x, y) \in \mathcal{U} \times \mathcal{V}, \tag{22}$$

*then $F + G$ is metrically regular around $(\bar{x}, \bar{y})$ with modulus $\tau$.*

*As a result, if $G$ is upper semicontinuous at $\bar{x}$ and $G(\bar{x}) = \{\bar{k}\}$, then $F + G$ is metrically regular around $(\bar{x}, \bar{y})$ with modulus $\tau$.*

*Proof* Suppose that (22) holds for every $(x, y) \in B(\bar{x}, \delta_1) \times B(\bar{y}, \delta_2)$ for some $\delta_1, \delta_2 > 0$. Since $(F, G)$ is locally sum-stable around $(\bar{x}, \bar{y} - \bar{k}, \bar{k})$, there exists $\delta > 0$ such that, for every $x \in B(\bar{x}, \delta)$ and every $w \in (F + G)(x) \cap B(\bar{y}, \delta)$, there are $y \in F(x) \cap B(\bar{y} - \bar{k}, \theta)$ and $z \in G(x) \cap B(\bar{k}, \theta)$ such that $w = y + z$. Taking $\delta$ smaller if necessary, we can assume that $\delta < \delta_1$. Fix $(x, y) \in B(\bar{x}, \delta/2) \times B(\bar{y}, \delta/2)$. We consider two following cases:

Case 1. $d(y, F(x) + G(x)) < \delta/2$. Fix $\gamma > 0$ , small enough in order to have

$$d(y, F(x) + G(x)) + \gamma < \delta/2,$$

and take $t \in F(x) + G(x)$ such that $\|y - t\| < d(y, F(x) + G(x)) + \gamma$. Hence we have $\|y - t\| < \delta/2$, and since we also have $\|y - \bar{y}\| < \delta/2$, this yields

$$\|t - \bar{y}\| \leq \|y - t\| + \|y - \bar{y}\| < \delta/2 + \delta/2 = \delta.$$

It follows that

$$t \in [F(x) + G(x)] \cap B(\bar{y}, \delta).$$

Since $(F, G)$ is locally sum-stable around $(\bar{x}, \bar{y}-\bar{k}, \bar{k})$, there are $y \in F(x)\cap B(\bar{y}-\bar{k}, \theta)$ and $z \in G(x)\cap B(\bar{k}, \theta)$ such that

$$t = y + z.$$

Consequently,

$$t \in F(x) \cap B(\bar{y} - \bar{k}, \theta) + G(x) \cap B(\bar{k}, \theta) \subset F(x) + G(x) \cap B(\bar{k}, \theta).$$

Therefore,

$$d(y, F(x) + G(x) \cap B(\bar{k}, \theta)) \leq \|y - t\|,$$

from which we derive

$$d(y, F(x) + G(x) \cap B(\bar{k}, \theta)) \leq d(y, F(x) + G(x)) + \gamma,$$

and therefore, as $\gamma$ is arbitrarily small, we obtain that

$$d(y, F(x) + G(x) \cap B(\bar{k}, \theta)) \leq d(y, F(x) + G(x)).$$

By (22), one gets that

$$d(x, (F + G)^{-1}(y)) \leq \tau d(y, F(x) + G(x)).$$

Since $(x, y)$ is arbitrary in $B(\bar{x}, \delta/2) \times B(\bar{y}, \delta/2)$, this yields

$$d(x, (F + G)^{-1}(y)) \leq \tau d(y, F(x) + G(x)),$$

for all $(x, y) \in B(\bar{x}, \delta/2) \times B(\bar{y}, \delta/2)$.

Case 2. If $d(y, F(x) + G(x)) \geq \delta/2$. Choose $\delta$ sufficiently small so that $\tau\delta/4 < \delta_1$. For every $(x, y) \in B(\bar{x}, \tau\delta/4) \times B(\bar{y}, \delta/4)$ and any $\varepsilon > 0$, by (22), there exists $u \in (F + G)^{-1}(y)$ such that

$$d(\bar{x}, u) < (1 + \varepsilon)\tau d(y, F(\bar{x}) + G(\bar{x})) \leq (1 + \varepsilon)\tau\|y - \bar{y}\| < (1 + \varepsilon)\tau\delta/2 \leq (1 + \varepsilon)\tau/2d(y, F(x) + G(x)).$$

So,

$$\begin{aligned} d(x,u) &\le d(x,\bar{x}) + d(\bar{x},u) \\ &< \tau\delta/4 + (1+\varepsilon)\tau/2d(y,F(x)+G(x)) \\ &< \tau/2d(y,F(x)+G(x)) + (1+\varepsilon)\tau/2d(y,F(x)+G(x)). \end{aligned}$$

Taking the limit as $\varepsilon > 0$ goes to 0, it follows that

$$d(x,(F+G)^{-1}(y)) \le \tau d(y,F(x)+G(x)).$$

So,

$$d(x,(F+G)^{-1}(y)) \le \tau d(y,F(x)+G(x)),$$

for all $(x,y) \in B(\bar{x},\tau\delta/4) \times B(\bar{y},\delta/4)$. The proof is complete. △

The following example shows that the sum of a metrically regular set-valued mapping and a pseudo-Lipschitz one is not generally metrically regular without the sum-stability (see [15] for a similar example on the sum of two pseudo-Lipschitz set-valued mappings).

*Example 3.1* Let $F, G : \mathbb{R} \rightrightarrows \mathbb{R}$ be given as

$$F(x) := \begin{cases} [-x,+\infty[, & \text{if } x \in [0,+\infty[ \\ \{-1\}, & \text{otherwise} \end{cases}$$

and

$$G(x) := \{0,1\},\ x \in \mathbb{R}.$$

Then, obviously, $F, G$ are closed multifunctions and, it is easy to see that $F$ is metrically regular around $(0,0)$ and $G$ is pseudo-Lipschitz around $(0,0)$. However, $(F,G)$ is not sum-stable around $(0,0,0)$ and $F+G$ fails to be metrically regular around $(0,0)$.

*Proof* Indeed, we have that

$$(F+G)(x) = \begin{cases} [-x,+\infty[, & \text{if} x \in [0,+\infty[ \\ +\infty & \text{otherwise.} \end{cases}$$

and

$$(F+G)^{-1}(x) = \begin{cases} ]-x,+\infty[, & \text{if } x \in ]-\infty,0[\setminus\{-1\} \\ \mathbb{R}, & \text{if } x = 0 \\ ]-\infty,0]\cup]1,+\infty[, & \text{if } x = -1 \\ ]0,+\infty[\cup\{1\}, & \text{if } x \in ]0,+\infty[. \end{cases}$$

Suppose that $F+G$ is metrically regular around $(0,0)$, then there exist $\tau > 0$ and $0 < \delta < \min\{1, \tau^{-1}\}$ such that, for every $(x,y) \in ]-\delta,\delta[\times]-\delta,\delta[$, one has

$$d(x,(F+G)^{-1}(y)) \leq \tau d(y,(F+G)(x)). \tag{23}$$

Consider $x := -\delta/2$ and $y := -\delta^2/2$. Then, $x \in ]-\infty, 0[$, $y \in ]-\infty, 0[$ and

$$(F+G)(x) = \{-1, 0\}, (F+G)^{-1}(y) = ]\delta^2/2, +\infty[.$$

Thus,

$$d(x,(F+G)^{-1}(y)) = d(-\delta/2, ]\delta^2/2, +\infty[) = |-\delta/2 - (\delta^2/2)| = \delta/2 + \delta^2/2,$$

and,

$$d(y,(F+G)(x)) = d(-\delta^2/2, \{-1,0\}) = \min\{1-\delta^2/2, \delta^2/2\} = \delta^2/2.$$

Consequently, by (23), one obtains that $\delta/2 + \delta^2/2 \leq \tau\delta^2/2$. Since, $1 < 1+\delta \leq \tau\delta$, this yields $\delta > \tau^{-1}$, which contradits the choice of $\delta$. Hence, $F+G$ can not metrically regular around $(0,0)$.
Of course, $(F,G)$ is not sum-stable around $(0,0,0)$. Indeed, take $0 < \varepsilon < 1$, then, for every $\delta > 0$, consider $x_\delta := \delta/2 \in ]-\delta,\delta[$ and $w_\delta := \delta/2 \in (F+G)(x_\delta) \cap ]-\delta,\delta[ = ]-\delta/2, \delta[$. By taking $\varepsilon$ smaller if necessary, we can assume that $\delta > 2\varepsilon$. Then, for every $y_\delta \in F(x_\delta) \cap (-\varepsilon,\varepsilon) = ]-\varepsilon,\varepsilon[$ and, for every $z_\delta \in G(x_\delta) \cap ]-\varepsilon,\varepsilon[ = \{0\}$, one has $w_\delta = \delta/2 > \varepsilon + 0 > y_\delta + z_\delta$. △

The following theorem establishes metric regularity of the multifunction $\mathcal{E}_{(F,G)}$ as well as metric regularity of the sum mapping, of course, with the sum-stable assumption added.

**Theorem 3.3** *Let $X$ be a complete metric space, let $Y$ be a Banach space and let $F, G : X \rightrightarrows Y$ be closed multifunctions. Suppose that $(\bar{x}, \bar{k}, \bar{y}) \in X \times Y \times Y$ is such that $\bar{y} \in F(\bar{x}) + \bar{k}, \bar{k} \in G(\bar{x})$, $F$ be metrically regular around $(\bar{x}, \bar{y} - \bar{k})$ with modulus $\tau > 0$ and $G$ is pseudo-Lipschitz around $(\bar{x}, \bar{k})$ with modulus $\lambda > 0$ with $\tau\lambda < 1$. Suppose that the product space $X \times Y$ is endowed with the metric defined by*

$$d((x,k),(u,z)) = \max\{d(x,u), \|z-k\|/\lambda\}.$$

*Then $\mathcal{E}_{(F,G)}$ is metrically regular around $(\bar{x}, \bar{k}, \bar{y})$ with modulus $(\tau^{-1} - \lambda)^{-1}$.*

*If in addition we suppose that the pair $(F,G)$ is locally sum-stable around $(\bar{x}, \bar{y} - \bar{k}, \bar{k})$, then $F+G$ is metrically regular around $(\bar{x}, \bar{y})$ with modulus $(\tau^{-1} - \lambda)^{-1}$.*

*Proof* Since by assumption $G$ is pseudo-Lipschitz around $(\bar{x}, \bar{k})$ with modulus $\lambda > 0$, there exist $\delta_1, \delta_2 > 0$ such that

$$G(x_1) \cap B(\bar{k}, \delta_1) \subset G(x_2) + \lambda\|x_1 - x_2\|\bar{B}_Y, \quad \text{for all } x_1, x_2 \in B(\bar{x}, \delta_2). \tag{24}$$

Then, obviously, $G$ is lower semicontinuous at all $(x,k) \in (B(\bar{x},\delta_2) \times B(\bar{k},\delta_1)) \cap \mathrm{gph}\, G$.. Therefore, $\varphi_{\mathcal{E}}$ is given by the second equality in Lemma 3.2. Furthermore, since $F$ is metrically regular around $(\bar{x}, \bar{y}-\bar{k})$ with modulus $\tau > 0$, there exist $\delta_3, \delta_4 > 0$ and a real $\gamma > 0$ such that

$$|\nabla \varphi_F(\cdot, y)|(x) \geq \frac{1}{\tau} \quad \text{for all } (x,y) \in B(\bar{x},\delta_3) \times B(\bar{y}-\bar{k},\delta_4) \quad \text{with } \varphi_F(x,y) \in ]0,\gamma[. \tag{25}$$

So, for any $\varepsilon > 0$, there exists $u \in B(x,\delta_3), u \neq x$ such that

$$\frac{\varphi_F(x,y) - \varphi_F(u,y)}{d(x,u)} > \frac{1}{\tau + \varepsilon/2}.$$

Taking $\delta_1, \delta_3$ smaller if neccesary, we can assume that $\delta_1 < \delta_4$, and $\delta_3 < \delta_2$. Then, for every $(x,k,y) \in B(\bar{x}, \min\{\delta_2,\delta_3\}/2) \times B(\bar{k},\delta_1) \times B(\bar{y}, \delta_4 - \delta_1)$ with $y - k \notin F(x), k \in G(x)$, any $\varepsilon > 0$ and any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converging to $x$, $\{k_n\}_{n\in\mathbb{N}} \subseteq X$ converging to $k$ with $k_n \in G(x_n)$, and

$$\lim_{n\to\infty} d(y-k_n, F(x_n)) = \liminf_{u\to x} d(y-k, F(u)),$$

we deduce that

$$\frac{\varphi_F(x,y-k) - \varphi_F(u,y-k)}{d(x,u)} > \frac{1}{\tau+\varepsilon/2}, (\text{since } y-k \in B(\bar{y}-\bar{k},\delta_4)), \tag{26}$$

and

$$\lim_{n\to\infty} d(y-k, F(x_n)) = \lim_{n\to\infty} d(y-k_n, F(x_n)) = \liminf_{u\to x} d(y-k, F(u)) = \varphi_F(x, y-k).$$

On the other hand, by definition of the function $\varphi_{\mathcal{E}}$, there is a sequence $\{u_n\}_{n\in\mathbb{N}} \subseteq X$ converging to $u$ such that

$$\lim_{n\to\infty} d(y-k, F(u_n)) = \varphi_F(u, y-k).$$

Because $u \in B(x,\delta_3), x \in B(\bar{x}, \min\{\delta_2,\delta_3\}/2), \{u_n\}_{n\in\mathbb{N}} \to u$, for $n$ large enough, one has that $u_n \in B(\bar{x},\delta_2)$. Similarly, since $k \in B(\bar{k},\delta_1)$ and $\{k_n\}_{n\in\mathbb{N}} \subseteq X$ converges to $k$, for $n$ large enough, one has that $k_n \in B(\bar{k},\delta_1)$.

Therefore, by (24), and (26), there exists $z_n \in G(u_n)$ such that

$$\|z_n - k_n\| \leq \lambda d(x_n, u_n). \tag{27}$$

and

$$\lim_{n\to\infty} \frac{d(y-k, F(x_n)) - d(y-k, F(u_n))}{d(x,u)} > \frac{1}{\tau+\varepsilon}.$$

Thus, noting that $u \neq x$, one has that

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{d(y-k, F(x_n)) - d(y-k, F(u_n))}{d(x_n, u_n)} \\
&= \lim_{n\to\infty} \frac{d(y-k, F(x_n)) - d(y-k, F(u_n))}{d(x_n, u_n)} \\
&= \lim_{n\to\infty} \frac{d(y-k, F(x_n)) - d(y-k, F(u_n))}{d(x, u)} \frac{d(x,u)}{d(x_n, u_n)} \\
&= \lim_{n\to\infty} \frac{d(y-k, F(x_n)) - d(y-k, F(u_n))}{d(x, u)} \lim_{n\to\infty} \frac{d(x,u)}{d(x_n, u_n)} \\
&\leq \frac{1}{\tau+\varepsilon}.
\end{aligned}$$

On the other hand,

$$d(y - z_n, F(u_n)) \leq d(y - k_n F(u_n)) + \|k_n - z_n\|. \tag{28}$$

From relations (27), (28), we deduce that for any $(x, k, y) \in B(\bar{x}, \min\{\delta_2, \delta_3\}/2) \times B(\bar{k}, \delta_1) \times B(\bar{y}, \delta_4 - \delta_1)$ with $y - k \notin F(x), k \in G(x)$, and any $\varepsilon > 0$, any sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ converging to $x$, $\{k_n\}_{n\in\mathbb{N}} \subseteq X$ converging to $k$, there exists $\{(u_n, z_n)\}_{n\in\mathbb{N}}$ with

$$\liminf_{n\to\infty} d((u_n, z_n), (x, k)) = \liminf_{n\to\infty} \max\{d(u_n, x), \|z_n - k\|/\lambda\} \geq \liminf_{n\to\infty} d(u_n, x) > 0,$$

(since $0 < d(x, u) \leq d(u_n, x) + d(u_n, u)$ and $u_n \to u$)

such that

$$\begin{aligned}
&\limsup_{n\to\infty} \frac{d(y-k_n, F(x_n)) - d(y-z_n, F(u_n))}{d((x_n,k_n),(u_n,z_n))} \\
&\geq \limsup_{n\to\infty} \frac{d(y-k_n, F(x_n)) - d(y-k_n, F(u_n)) - \|k_n - z_n\|}{d((x_n,k_n),(u_n,z_n))} \\
&= \limsup_{n\to\infty} \frac{d(y-k_n, F(x_n)) - d(y-k_n, F(u_n)) - \|k_n - z_n\|}{\max\{d(x_n,u_n), \|k_n - z_n\|/\lambda\}} \\
&\geq \limsup_{n\to\infty} \frac{d(y-k_n, F(x_n)) - d(y-k_n, F(u_n))}{\max\{d(x_n,u_n), \|k_n - z_n\|/\lambda\}} - \lambda \\
&= \limsup_{n\to\infty} \frac{d(y-k_n, F(x_n)) - d(y-k_n, F(u_n))}{d(x_n,u_n)} - \lambda > \frac{1}{\tau+\varepsilon} - \lambda,
\end{aligned}$$

(since $\|z_n - k_n\|/\lambda \leq d(x_n, u_n)$).

By Lemma 3.4 ($(i) \Leftrightarrow (ii)$), one concludes that $\mathcal{E}_{(F,G)}$ is metrically regular around $(\bar{x}, \bar{k}, \bar{y})$ with modulus $(\tau^{-1} - \lambda)^{-1}$.

If the pair $(F, G)$ is locally sum-stable around $(\bar{x}, \bar{y} - \bar{k}, \bar{k})$, then, combining the hypothesis Proposition 3.3 and Proposition 3.1, we complete the proof. △

Combining Proposition 3.1 and Theorem 3.3, we obtain the following corollary, which is equivalent to the main result (Theorem 3.3) in [15], which is stated for the difference of an open mapping and a pseudo-Lipschitz one.

**Corollary 3.1** *Let $X$ be a complete metric space, let $Y$ be a Banach space and let $F, G : X \rightrightarrows Y$ be closed multifunctions. Suppose that $(\bar{x}, \bar{k}, \bar{y}) \in X \times Y \times Y$ is such that $\bar{y} \in F(\bar{x}) + \bar{k}, \bar{k} \in G(\bar{x})$ and $F$ is metrically regular around $(\bar{x}, \bar{y} - \bar{k})$ with modulus $\tau > 0$ and $G$ is pseudo-Lipschitz around $(\bar{x}, \bar{k})$ with modulus $\lambda > 0$ with $\tau\lambda < 1$. Then, there exist a neighborhood $\mathcal{U} \times \mathcal{V} \times \mathcal{W}$ of $(\bar{x}, \bar{k}, \bar{y} - \bar{k})$ and $\varepsilon, \tau > 0$ such that, for every $(x, k, z) \in \mathcal{U} \times \mathcal{V} \times \mathcal{W}, k \in G(x), z \in F(x)$, and $\rho \in ]0, \varepsilon[$,*

$$B(k + z, \rho\tau^{-1}) \subset (F + G)(B(x, \rho)).$$

## 4 Metric Regularity of the Epigraphical Multifunction under Coderivative Conditions

In this section, $X, Y$ are assumed to be Asplund spaces, i.e., Banach spaces for which each separable subspace has a separable dual (in particular, any reflexive space is Asplund; see, e.g., [20, 22] for more details). We recall some notation, terminology and definitions basically standard and conventional in the area of variational analysis and generalized differentials (see [20, 22–25, 61] and the references therein). As usual, $\|\cdot\|$ stands for the norm on $X$ or $Y$, indifferently, and $\langle\cdot,\cdot\rangle$ signifies for the canonical pairing between $X$ and its topological dual $X^\star$ with the symbol $\xrightarrow{w^\star}$ indicating the convergence in the weak$^\star$ topology of $X^\star$ and the symbol $\mathrm{cl}^*$ standing for the weak$^\star$ topological closure of a set. Given a set-valued mapping $F\colon X \rightrightarrows X^\star$ between $X$ and $X^\star$, recall that the symbol

$$\operatorname*{Lim\,sup}_{x\to\bar{x}} F(x) := \left\{ x^\star \in X^\star \,\middle|\, \exists x_n \to \bar{x},\ \exists x_n^\star \xrightarrow{w^\star} x^\star \ \text{ with } \ x_n^\star \in F(x_n), \quad n \in \mathbb{N} \right\} \tag{29}$$

stands for the *sequential Painlevé-Kuratowski outer/upper limit* of $F$ as $x \to \bar{x}$ with respect to the norm topology of $X$ and the weak$^\star$ topology of $X^\star$. Let us consider $f : X \to \mathbb{R} \cup \{+\infty\}$ an extended-real-valued lower semicontinuous function and $\bar{x}$ fixed in $X$. The notation $x \xrightarrow{f} \bar{x}$ means that with $x \to \bar{x}$ with $f(x) \to f(\bar{x})$. The Fréchet subdifferential $\hat{\partial} f(\bar{x})$ of $f$ at $\bar{x}$ is given by the formula:

$$\hat{\partial} f(\bar{x}) = \left\{ x^\star \in X^\star : \liminf_{x\to\bar{x},\, x\neq\bar{x}} \frac{f(x) - f(\bar{x}) - \langle x^\star, x - \bar{x}\rangle}{\|x - \bar{x}\|} \geq 0 \right\},$$

and $\hat{\partial} f(\bar{x}) = \emptyset$ if $\bar{x} \notin \operatorname{Dom} f$.

The notation $\partial f(\bar{x})$ is used to denote the limiting subdifferential of $f$ at $\bar{x} \in \operatorname{Dom} f$. It is defined by

$$\partial f(\bar{x}) := \operatorname*{Lim\,sup}_{x \xrightarrow{f} \bar{x}} \hat{\partial} f(x).$$

For a closed set $C \subset X$ and $\bar{x} \in C$, the Fréchet normal cone to $C$ at $\bar{x}$ is denoted $\hat{N}(\bar{x}; C)$ and is defined as the Fréchet subdifferential of indicator function $\delta_C$ of $C$ at $\bar{x}$, i.e.,

$$\hat{N}(\bar{x}; C) := \hat{\partial}\delta_C(\bar{x}),$$

where $\delta_C(x) = 0$ if $x \in C$, and $\delta_C(x) = +\infty$ if $x \notin C$.

The limiting normal cone of $C$ at $\bar{x}$ is defined and denoted by

$$N(\bar{x}; C) = \partial\delta_C(\bar{x}).$$

Let us consider a closed multifunction $F : X \rightrightarrows Y$ and $\bar{y} \in F(\bar{x})$. The Fréchet coderivative of $F$ at $(\bar{x}, \bar{y})$ is the mapping $\hat{D}^\star F(\bar{x}, \bar{y}) : Y^\star \rightrightarrows X^\star$ defined by

$$x^\star \in \hat{D}^\star F(\bar{x}, \bar{y})(y^\star) \Leftrightarrow (x^\star, -y^\star) \in \hat{N}((\bar{x}, \bar{y}); \text{gph } F),$$

while the Mordukhovich (limiting) coderivative of $F$ at $(\bar{x}, \bar{y})$ is the mapping $D^\star F(\bar{x}, \bar{y}) : Y^\star \rightrightarrows X^\star$ defined by

$$x^\star \in D^\star F(\bar{x}, \bar{y})(y^\star) \Leftrightarrow (x^\star, -y^\star) \in N((\bar{x}, \bar{y}); \text{gph } F).$$

Here, $\hat{N}((\bar{x}, \bar{y}); \text{gph } F)$ and $N((\bar{x}, \bar{y}); \text{gph } F)$ are the Fréchet and the limiting normal cone to gph $F$ at $(\bar{x}, \bar{y})$, respectively.

To obtain a point-based condition for metric regularity of multifunctions in infinite dimensional spaces, one often uses the so-called *partial sequential normal compactness* (PSNC) property.

A multifunction $F : X \rightrightarrows Y$ is *partially sequentially normally compact* at $(\bar{x}, \bar{y}) \in$ gph $F$, iff, for any sequences $\{(x_k, y_k, x_k^\star, y_k^\star)\} \in$ gph $F \times X^\star \times Y^\star$ satisfying

$$(x_k, y_k) \to (\bar{x}, \bar{y}), x_k^\star \in \hat{D}^\star(x_k, y_k)(y_k^\star), x_k^\star \xrightarrow{w^\star} 0, \|y_k^\star\| \to 0,$$

one has $\|x_k^\star\| \to 0$ as $k \to \infty$.

*Remark 4.1* Condition (PSNC) at $(\bar{x}, \bar{y}) \in$ gph $F$ is satisfied if $X$ is finite dimensional, or $F$ is pseudo-Lipschitz around that point.

In the following, we need a result on *the metric inequality* (see, e.g., Ioffe [3], Huynh &Théra [57]). Let us recall that the sets $\{\Omega_1, \Omega_2\}$ satisfy the metric inequality at $\bar{x}$ iff, there are $\tau > 0$ and $r > 0$ such that

$$d(x, \Omega_1 \cap \Omega_2) \leq \tau[d(x, \Omega_1) + d(x, \Omega_2)] \text{ for all } x \in B(\bar{x}, r).$$

**Definition 4.1** We say that at $\bar{x}$, property $(\mathcal{H})$ is satisfied if

for any sequences $\{x_{ik}\}_{k\in\mathbb{N}} \subset \Omega_i$ $(i = 1, 2)$, $\{x_{ik}{}^*\} \in \hat{N}(x_{ik}; \Omega_i)_{k\in\mathbb{N}}$ $(i = 1, 2)$ such that $\{x_{ik}\}_{k\in\mathbb{N}} \to \bar{x}$, and $\|x_{1k}^\star + x_{2k}^\star\|_{k\in\mathbb{N}} \to 0$, then necessarily $\{x_{1k}^\star\} \to 0$ and $\{x_{2k}^\star\} \to 0$.

Property $(\mathcal{H})$ was called by A. Y. Kruger, dual (or normal) uniform regularity (see, [55] and [56] for a comparison between hypothesis $(\mathcal{H})$ and the metric inequality. One can also note that $(\mathcal{H})$ is the Asplund space version of the Mordukhovich "limiting qualification condition" (cf. [22, Definition 3.2 (ii)]). Although formally the last one is weaker, it is easy to show that in the Asplund space setting the two conditions are equivalent.

**Proposition 4.1** *Let $\{\Omega_1, \Omega_2\}$ be two closed subsets of $X$ and fix $\bar{x} \in \Omega_1 \cap \Omega_2$. If we suppose that property $(\mathcal{H})$ holds, then the sets $\{\Omega_1, \Omega_2\}$ satisfy the metric inequality at $\bar{x}$. Under this assumption, there is some $r > 0$ such that for every $\varepsilon > 0$, and $x \in B(\bar{x}, r)$, there exist $x_1, x_2 \in B(x, \varepsilon)$ such that*

$$\hat{N}(x; \Omega_1 \cap \Omega_2) \subset \hat{N}(x_1; \Omega_1) + \hat{N}(x_2; \Omega_2) + \varepsilon B_{X^\star}. \tag{30}$$

Let us consider two multifunctions $F, G : X \rightrightarrows Y$. To these multifunctions, we associate the two sets

$$C_1 := \{(x, y, z) \in X \times Y \times Y : y \in G(x)\} \text{ and } C_2 := \{(x, y, z) \in X \times Y \times Y : z \in F(x)\}.$$

*Remark 4.2* Hypothesis $(\mathcal{H})$ can be restated for the sets $\{C_1, C_2\}$ at $(\bar{x}, \bar{y}, \bar{z}) \in C_1 \cap C_2$ as follows:

(i) $(\mathcal{H})$: for any sequences

$$\{(x_k, y_k)\}_{k \in \mathbb{N}} \subset \operatorname{gph} G, \{(v_k, z_k)\}_{k \in \mathbb{N}} \subset \operatorname{gph} F,$$

$$x_k^\star \in \hat{D}^\star G(x_k, y_k)(y_k^\star), u_k^\star \in \hat{D}^\star F(v_k, z_k)(z_k^\star),$$

such that if

$$(x_k, y_k) \to (\bar{x}, \bar{y}),$$

$$(v_k, z_k) \to (\bar{x}, \bar{z}),$$

$$\|x_k^\star + u_k^\star\| \to 0,$$

$$y_k^\star \to 0, z_k^\star \to 0,$$

then

$$x_k^\star \to 0, u_k^\star \to 0, \text{as } k \to 0.$$

It holds whenever one of following conditions is fulfilled:

(ii) $F^{-1}$ or $G^{-1}$ is pseudo-Lipschitz around $(\bar{z}, \bar{x})$ or $(\bar{y}, \bar{x})$, respectively;

(iii) either $F$ is PSNC at $(\bar{x}, \bar{z})$ or $G$ is PSNC at $(\bar{x}, \bar{y})$, and

$$D^\star F(\bar{x}, \bar{z})(0) \cap -D^\star G(\bar{x}, \bar{y})(0) = \{0\}.$$

*Proof* Observe that, if $F^{-1}$ or $G^{-1}$ is pseudo-Lipschitz around $(\bar{z}, \bar{x})$ and $(\bar{y}, \bar{x})$, respectively, then assumption $(\mathcal{H})$ always holds (see for instance [22]).

We now assume that (iii) holds. Take

$$\{(x_k, y_k)\}_{k \in \mathbb{N}} \subset \operatorname{gph} G, \{(v_k, z_k)\}_{k \in \mathbb{N}} \subset \operatorname{gph} F,$$

$$x_k^\star \in \hat{D}^\star G(x_k, y_k)(y_k^\star), u_k^\star \in \hat{D}^\star F(v_k, z_k)(z_k^\star),$$

such that

$$(x_k, y_k) \to (\bar{x}, \bar{y}),$$

$$(v_k, z_k) \to (\bar{x}, \bar{z}),$$

$$\|x_k^\star + u_k^\star\| \to 0,$$

$$y_k^\star \to 0, z_k^\star \to 0.$$

If the sequences $\{x_k^\star\}_{k\in\mathbb{N}}$, $\{u_k^\star\}_{k\in\mathbb{N}}$ are unbounded, we can assume that

$$\|x_k^\star\| \to \infty, \|u_k^\star\| \to \infty,$$

and

$$\frac{x_k^\star}{\|x_k^\star\|} \xrightarrow{w^\star} x^\star, \frac{u_k^\star}{\|u_k^\star\|} \xrightarrow{w^\star} u^\star.$$

Then,

$$y_k^\star/\|x_k^\star\| \to 0 \text{ and } z_k^\star/\|u_k^\star\| \to 0.$$

Consequently,

$$x^\star \in D^\star G(\bar{x}, \bar{y})(0), u^\star \in D^\star F(\bar{x}, \bar{z})(0).$$

On the other hand,

$$u^\star + x^\star = 0, \text{ (since } \|x_k^\star + u_k^\star\| \to 0).$$

It follows that

$$u^\star \in D^\star F(\bar{x}, \bar{z})(0) \cap -D^\star G(\bar{x}, \bar{y})(0).$$

Therefore, by assumption, this yields $x^\star = u^\star = 0$.

Hence,

$$\frac{x_k^\star}{\|x_k^\star\|} \to 0, \quad \text{or} \quad \frac{u_k^\star}{\|u_k^\star\|} \to 0, \text{ (by PSNC property of } F \text{ or } G).$$

This contradicts the fact that $\frac{x_k^\star}{\|x_k^\star\|}$, and $\frac{u_k^\star}{\|u_k^\star\|}$ are in the unit sphere $S_{Y^\star}$ of $Y^\star$. So, the sequences $\{x_k^\star\}_{k\in\mathbb{N}}$, $\{u_k^\star\}_{k\in\mathbb{N}}$ are bounded. Without any loss of generality, we can assume that

$$x_k^\star \xrightarrow{w^\star} x^\star, u_k^\star \xrightarrow{w^\star} u^\star.$$

It follows that

$$x^\star \in D^\star G(\bar{x}, \bar{y})(0), u^\star \in D^\star F(\bar{x}, \bar{z})(0).$$

Moreover,

$$x^\star + u^\star = 0.$$

Hence,

$$u^\star \in D^\star F(\bar{x}, \bar{z})(0) \cap -D^\star G(\bar{x}, \bar{y})(0).$$

Therefore, by assumption, we obtain $x^\star = u^\star = 0$, and $x_k^\star \to 0$, or, $u_k^\star \to 0$, (by PSNC property of $F$ or $G$). The proof is complete. △

The following lemma gives an estimation for the strong slope of the function $\varphi_{\mathcal{E}}((x,k),y)$.

**Lemma 4.1** *Let $(\bar{x},\bar{y}-\bar{k},\bar{k}) \in X \times F(\bar{x}) \times G(\bar{x})$ be given. Assume that the sets $\{C_1, C_2\}$ defined, as above, satisfy hypothesis $(\mathcal{H})$ at $(\bar{x},\bar{k},\bar{y}-\bar{k})$. Then there exists $\rho > 0$ such that, for all $(x,k,y) \in B((\bar{x},\bar{k},\bar{y}),\rho)$ with $y \notin F(x)+k, k \in G(x)$ as well as $d(y, F(x)+k) < \rho$, one has*

$$|\nabla \varphi_{\mathcal{E}}((\cdot,\cdot),y)|(x,k) \geq \lim_{\delta \downarrow 0} \left\{ \inf \left\{ \|x^\star\| : \begin{array}{l} (u,w) \in \text{gph } F, (v,z) \in \text{gph } G, u, v \in B(x,\delta), \\ u^\star \in \hat{D}^\star G(v,z)(y^\star), \|y^\star\| = 1, z \in B(k,\delta) \\ x^\star \in \hat{D}^\star F(u,w)(y^\star + z^\star) + u^\star, z^\star \in \delta B_{Y^\star}, \\ |\|w+k-y\| - \varphi_{\mathcal{E}}((x,k),y)| < \delta, \\ |\langle y^\star + z^\star, w+k-y \rangle - \|w+k-y\|| < \delta \end{array} \right\} \right\}.$$

*Proof* Obviously, if $(\mathcal{H})$ is satisfied at $(\bar{x},\bar{k},\bar{y}-\bar{k})$ then it is also satisfied at all points $(u,v,w) \in X \times G(u) \times F(u)$ near $(\bar{x},\bar{k},\bar{y}-\bar{k})$, say $(u,v,w) \in X \times G(u) \times F(v) \cap B_{X\times Y\times Y}((\bar{x},\bar{k},\bar{y}-\bar{k}),3\rho)$. Let $(x,k,y) \in B_{X\times Y\times Y}((\bar{x},\bar{k},\bar{y}-\bar{k}),\rho)$ be such that $y \notin F(x)+k, k \in G(x)$ and $d(y,F(x)+k) < \rho$. Set $|\nabla\varphi_{\mathcal{E}}((\cdot,\cdot),y)|(x,k) := m$. By the lower semicontinuity of $\varphi_{\mathcal{E}}$ (Note that $\varphi_{\mathcal{E}}$ is given by the first equality in Lemma 3.2) as well as the definition of the strong slope, for each $\varepsilon \in ]0, \varphi_{\mathcal{E}}((x,k),y)[$, there is $\eta \in (0,\varepsilon)$ with $4\eta + \varepsilon < \varphi_{\mathcal{E}}((x,k),y)$ and $1-(m+\varepsilon+3)\eta > 0$ such that $d(y, F(u)+l) \geq \varphi_{\mathcal{E}}((x,k),y) - \varepsilon,$ for all $u \in B(x,4\eta)$, $l \in B(k,\eta) \cap G(u)$ and

$$m + \varepsilon \geq \frac{\varphi_{\mathcal{E}}((x,k),y) - \varphi_{\mathcal{E}}((z,k'),y)}{\max\{\|x-z\|, \|k-k'\|\}} \quad \text{for all } z \in \bar{B}(x,\eta), k' \in \bar{B}(k,\eta) \cap G(x).$$

Consequently,

$$\varphi_{\mathcal{E}}((x,k),y) \leq \varphi_{\mathcal{E}}((z,k'),y) + (m+\varepsilon)\|z-x\| + (m+\varepsilon)\|k-k'\| \quad \text{for all } z \in \bar{B}(x,\eta), k' \in \bar{B}(k,\eta) \cap G(x).$$

By the definition of $\varphi_{\mathcal{E}}$, take $u \in B(x,\eta^2/4),\ v \in F(u), l \in B(k,\eta^2/8) \cap G(u)$ such that

$$\|y-l-v\| \leq \varphi_{\mathcal{E}}((x,k),y) + \eta^2/8.$$

By this way,

$$\|y-k-v\| \leq \varphi_{\mathcal{E}}((x,k),y) + \eta^2/4.$$

Taking into account that $\varphi_{\mathcal{E}}((z,k'),y) \leq d(y,F(z)+k')$ with $k' \in G(z)$, then $\varphi_{\mathcal{E}}((z,k'),y) \leq \|y-k'-w\|$ with $w \in F(z)$ and $k' \in G(z)$. It follows that

$$\varphi_{\mathcal{E}}((z,k'),y) \leq \|y-k'-w\| + \delta_{C_2}(z,k',w) + \delta_{C_1}(z,k',w).$$

From the inequality,

$$\|y-k-v\| \leq \varphi_{\mathcal{E}}((z,k'),y) + (m+\varepsilon)\|z-x\| + \eta^2/4,$$

we obtain that

$$\|y-k-v\| \leq \|y-k'-w\| + \delta_{C_2}(z,k',w) + \delta_{C_1}(z,k',w) + (m+\varepsilon)\|z-u\| + (m+\varepsilon)\eta + \eta^2/4,$$

for all $(z,w) \in \bar{B}(x,\eta) \times Y, k' \in \bar{B}(k,\eta)$. Applying the Ekeland variational principle [58] to the function

$$(z,k',w) \mapsto \|y-k'-w\| + \delta_{C_2}(z,k',w) + \delta_{C_1}(z,k',w) + (m+\varepsilon)\|z-u\|$$

on $\bar{B}(x,\eta) \times \bar{B}(k,\eta) \times Y$, we can select $(u_1,k_1,w_1) \in (u,k,v) + \frac{\eta}{4}B_{X\times Y\times Y}$ with $(u_1,k_1,w_1) \in C_2 \cap C_1$ such that

$$\|y-k_1-w_1\| \le \|y-k-v\| (\le \varphi_{\mathcal{E}}((x,k),y) + \eta^2/4); \tag{31}$$

and the function

$$(z,k',w) \mapsto \|y-k'-w\| + \delta_{C_2}(z,k',w) + \delta_{C_1}(z,k',w) + (m+\varepsilon)\|z-u\| + (m+\varepsilon+1)\eta\|(z,k',w)-(u_1,v_1,w_1)\|$$

attains a minimum on $\bar{B}(x,\eta) \times \bar{B}(k,\eta) \times Y$ at $(u_1,k_1,w_1)$. Hence, using the sum rule for Fréchet subdifferentials, we can find

$$(u_2,k_2,w_2),(u_4,k_4,w_4) \in B_{X\times Y\times Y}((u_1,k_1,w_1),\eta);\ (u_3,k_3,w_3) \in B_{X\times Y\times Y}((u_1,k_1,w_1),\eta) \cap C_2 \cap C_1;$$

such that

$$\begin{aligned} (0,k_2^\star,w_2^\star) &\in \hat{\partial}\|y-\cdot-\cdot\|(u_2,k_2,w_2), \\ (u_3^\star,k_3^\star,w_3^\star) &\in \hat{\partial}(\delta_{C_2}(\cdot,\cdot,\cdot)+\delta_{C_1}(\cdot,\cdot,\cdot))(u_3,k_3,w_3), \\ (u_4^\star,0,0) &\in \hat{\partial}((m+\varepsilon)\|\cdot-u\|)(u_4,k_4,w_4) \end{aligned}$$

and

$$(0,0,0) \in (0,k_2^\star,w_2^\star) + (u_3^\star,k_3^\star,w_3^\star) + (u_4^\star,0,0) + (m+\varepsilon+2)\eta[\bar{B}_{X^\star} \times \bar{B}_{Y^\star} \times \bar{B}_{Y^\star}]. \tag{32}$$

Note that

$$\begin{aligned} \|y-k_2-w_2\| &\ge \|y-v-k\| - \|w_2-v\| - \|k_2-k\| \\ &\ge \varphi_{\mathcal{E}}((x,k),y) - \varepsilon - (\|w_2-w_1\| + \|w_1-v\|) - (\|k_2-k_1\| + \|k-k_1\|) \\ &> \varphi_{\mathcal{E}}((x,k),y) - \varepsilon - 2\eta - 2\eta = \varphi_{\mathcal{E}}((x,k),y) - \varepsilon - 4\eta > 0. \end{aligned} \tag{33}$$

Then, by [59, Theorem 2.8.3] (see, also [60, proof of Theorem 3.6]), we know that

$$\hat{\partial}\|y-\cdot-\cdot\|(u_2,k_2,w_2) = \{(0,y^\star,y^\star) : y^\star \in S_{Y^\star}, \langle y^\star, w_2+k_2-y\rangle = \|y-w_2-k_2\|\}.$$

Hence,

$$w_2^\star = k_2^\star \in S_{Y^\star} \text{ and } \langle w_2^\star, w_2+k_2-y\rangle = \|y-w_2-k_2\|. \tag{34}$$

Now, in order to have $(u_3,k_3,w_3) \in B_{X\times Y\times Y}((\bar{x},\bar{k},\bar{y}-\bar{k}),3\rho)$, we take $\eta$ smaller if necessary, and, by virtue of Proposition 4.1, one has

$$(u_3^\star,k_3^\star,w_3^\star) \in \hat{N}((u_5,k_5,w_5);C_2) + \hat{N}((u_6,k_6,w_6);C_1) + \eta[\bar{B}_{X^\star} \times \bar{B}_{Y^\star} \times \bar{B}_{Y^\star}], \tag{35}$$

where $(u_5, k_5, w_5) \in C_2 \cap B_{X\times Y\times Y}((u_3, k_3, w_3), \eta), (u_6, k_6, w_6) \in C_1 \cap B_{X\times Y\times Y}((u_3, k_3, w_3), \eta)$. From (32) and (35), one deduces that

$$(0,0,0) \in (0, k_2^\star, w_2^\star) + \hat{N}((u_5, k_5, w_5); C_2)+$$

$$\hat{N}((u_6, k_6, w_6); C_1) + (u_4^\star, 0, 0) + (m+\varepsilon+3)\eta[\bar{B}_{X^\star} \times \bar{B}_{Y^\star} \times \bar{B}_{Y^\star}].$$

Therefore, there exist $(u_5^\star, k_5^\star, w_5^\star) \in [\bar{B}_{X^\star} \times \bar{B}_{Y^\star} \times \bar{B}_{Y^\star}], (u_6^\star, k_6^\star, 0) \in \hat{N}((u_6, k_6, w_6); C_1)$, i.e., $u_6^\star \in \hat{D}^\star G(u_6, k_6)(-k_6^\star)$ such that

$$(-u_4^\star - (m+\varepsilon+3)\eta u_5^\star - u_6^\star, -k_2^\star - (m+\varepsilon+3)\eta k_5^\star - k_6^\star, -w_2^\star - (m+\varepsilon+3)\eta w_5^\star) \in \hat{N}((u_5, k_5, w_5); C_2).$$

It follows that

$$-k_2^\star - (m+\varepsilon+3)\eta k_5^\star - k_6^\star = 0,$$

and

$$(-u_4^\star - (m+\varepsilon+3)\eta u_5^\star - u_6^\star, -w_2^\star - (m+\varepsilon+3)\eta w_5^\star) \in \hat{N}((u_5, w_5); \text{gph } F).$$

Consequently,

$$-k_6^\star = k_2^\star + (m+\varepsilon+3)\eta k_5^\star \text{ and } (-u_4^\star - (m+\varepsilon+3)\eta u_5^\star - u_6^\star) \in \hat{D}^\star F(u_5, w_5)(w_2^\star + (m+\varepsilon+3)\eta w_5^\star).$$

Remark that $\|k_6^\star\| = \| - k_2^\star - (m+\varepsilon+3)\eta k_5^\star\| \geq 1 - (m+\varepsilon+3)\eta > 0$.

Hence, setting

$$\begin{aligned}
y^\star &:= (k_2^\star + (m+\varepsilon+3)\eta k_5^\star)/\|k_2^\star + (m+\varepsilon+3)\eta k_5^\star\|,\\
z^\star &:= (w_5^\star - k_5^\star)(m+\varepsilon+3)\eta/\|k_2^\star + (m+\varepsilon+3)\eta k_5^\star\|,\\
x_1^\star &:= u_6^\star/\|k_2^\star + (m+\varepsilon+3)\eta k_5^\star\|,\\
x_2^\star &:= (-u_4^\star - (m+\varepsilon+3)\eta u_5^\star)/\|k_2^\star + (m+\varepsilon+3)\eta k_5^\star\|,
\end{aligned}$$

one obtains that

$$x_1^\star \in \hat{D}^\star G(u_6, k_6)(y^\star) \text{ and } (x_2^\star - x_1^\star) \in \hat{D}^\star F(u_5, w_5)(y^\star + z^\star), \tag{36}$$

where

$$\|y^\star\| = 1, \|z^\star\| \leq \frac{2(m+\varepsilon+3)\eta}{1-(m+\varepsilon+3)\eta} := \delta, \|x_2^\star\| \leq \frac{m+\varepsilon+(m+\varepsilon+3)\eta}{1-(m+\varepsilon+3)\eta}. \tag{37}$$

On the other hand, since $k_1 \in B(k, \eta), w_5 \in B(w_1, 2\eta)$, according to relation (31) one has

(38)

$$
\begin{aligned}
&\varphi_{\mathcal{E}}((x,k),y) - \varepsilon - 3\eta \\
&\qquad\qquad \leq \|y-k_1-w_1\| - \|w_5-w_1\| - \|k_1-k\| \leq \|y-k-w_5\| \\
&\qquad\qquad \leq \|y-k_1-w_1\| + \|w_5-w_1\| + \|k_1-k\| \\
&\qquad\qquad \leq \|y-k-v\| + 3\eta \\
&\qquad\qquad \leq \varphi_{\mathcal{E}}((x,k),y) + \eta^2/4 + 3\eta.
\end{aligned}
$$

Consequently,

$$
|\|y-k-w_5\| - \varphi_{\mathcal{E}}((x,k),y)| \leq 3\eta + \varepsilon. \tag{39}
$$

On the other hand, one has

$$
\langle y^\star + z^\star, y-k-w_5\rangle - \|y-k-w_5\| \leq \delta\|y-k-w_5\|; \tag{40}
$$

and, by $k_2 \in B(k,\eta)$; $w_2, w_5 \in B(w_1, 2\eta)$, from (34), one has the following estimates

$$
\begin{aligned}
&\langle y^\star + z^\star, k+w_5-y\rangle \\
&= \frac{\langle w_2^\star + (m+\varepsilon+3)\eta w_5^\star, k+w_5-y\rangle}{\|k_2^\star + (m+\varepsilon+3)\eta k_5^\star\|} \\
&= \frac{\langle w_2^\star, w_2+k_2-y\rangle + \langle w_2^\star, w_5-w_2\rangle + \langle w_2^\star, k-k_2\rangle + (m+\varepsilon+3)\eta\langle w_5^\star, w_5+k-y\rangle}{\|k_2^\star + (m+\varepsilon+3)\eta k_5^\star\|} \\
&\geq \frac{\|w_2+k_2-y\| - 3\eta - 2\eta - (m+\varepsilon+3)\eta\|w_5+k-y\|}{(1+(m+\varepsilon+3)\eta)}, \\
&\geq \frac{\|w_5+k-y\|(1-(m+\varepsilon+3)\eta) - 8\eta}{(1+(m+\varepsilon+3)\eta)}
\end{aligned} \tag{41}
$$

As $\varepsilon, \eta > 0$ are arbitrary small, by combining relations (36)-(41), we complete the proof.

**Theorem 4.1** *Let $X, Y$ be Asplund spaces, and let $F, G : X \rightrightarrows Y$ be closed multifunctions. Suppose that $(\bar{x}, \bar{k}, \bar{y}) \in X \times Y \times Y$ be such that $\bar{y} \in F(\bar{x}) + \bar{k}, \bar{k} \in G(\bar{x})$ and the sets $\{C_1, C_2\}$ satisfy the hypothesis $(\mathcal{H})$ at $(\bar{x}, \bar{k}, \bar{y} - \bar{k})$. Let $m > 0$. If there exist a neighborhood $\mathcal{U} \times \mathcal{V} \times \mathcal{W}$ of $(\bar{x}, \bar{k}, \bar{y})$ and $\gamma > 0$ such that, for each $(x, y, k) \in \mathcal{U} \times \mathcal{V} \times \mathcal{W}$ with $y \notin F(x) + k, k \in G(x)$,*

$$
m \leq \lim_{\delta \downarrow 0} \left\{ \inf \left\{ \|x^\star\| : \begin{array}{l} (u,w) \in \operatorname{gph} F, (v,z) \in \operatorname{gph} G, u, v \in B(x,\delta), \\ u^\star \in \hat{D}^\star G(v,z)(y^\star), \|y^\star\| = 1, z \in B(k,\delta), \\ x^\star \in \hat{D}^\star F(u,w)(y^\star + z^\star) + u^\star, z^\star \in \delta B_{Y^\star}, \\ \|w+k-y\| \leq \gamma + \delta, \\ |\langle y^\star + z^\star, w+k-y\rangle - \|w+k-y\|| < \delta \end{array} \right\} \right\},
$$

*then there exists a neighborhood $\mathcal{U}_1 \times \mathcal{V}_1 \times \mathcal{W}_1$ of $(\bar{x}, \bar{k}, \bar{y})$ such that*

$$
md((x,k), \mathbb{S}_{\mathcal{E}_{(F,G)}}(y)) \leq \varphi_{\mathcal{E}}((x,k),y) \quad \textit{for all} \quad (x,k,y) \in \mathcal{U}_1 \times \mathcal{V}_1 \times \mathcal{W}_1.
$$

This theorem implies the following result:

**Theorem 4.2** *Let $X, Y$ be Asplund spaces, and let $F, G : X \rightrightarrows Y$ be closed multifunctions, and let $(\bar{x}, \bar{k}, \bar{y}) \in X \times Y \times Y$ be such that $\bar{y} \in F(\bar{x}) + \bar{k}, \bar{k} \in G(\bar{x})$. Let $m > 0$. If the sets $\{C_1, C_2\}$ satisfy the hypothesis $(\mathcal{H})$ at $(\bar{x}, \bar{k}, \bar{y} - \bar{k})$ and*

$$m < \liminf_{(x_1,w)\xrightarrow{F}(\bar{x},\bar{y}-\bar{k}),(x_2,z)\xrightarrow{G}(\bar{x},\bar{k}),\delta\downarrow 0} \left\{ \|x^\star\| : \begin{array}{l} x^\star \in \hat{D}^\star F(x_1, w)(y^\star + \delta B_{Y^\star}) + u^\star \\ u^\star \in \hat{D}^\star G(x_2, z)(y^\star), \|y^\star\| = 1, \end{array} \right\} \tag{42}$$

*where the notations $(x_1, w) \xrightarrow{F} (\bar{x}, \bar{y} - \bar{k})$, $(x_2, z) \xrightarrow{G} (\bar{x}, \bar{k})$ mean that*

$$(x_1, w) \to (\bar{x}, \bar{y} - \bar{k}), (x_2, z) \to (\bar{x}, \bar{k}) \text{ and } (x_1, w) \in \text{gph } F, (x_2, z) \in \text{gph } G,$$

*then there exists a neighborhood $\mathcal{U}_1 \times \mathcal{V}_1 \times \mathcal{W}_1$ of $(\bar{x}, \bar{k}, \bar{y})$ such that*

$$md((x, k), \mathbb{S}_{\mathcal{E}_{(F,G)}}(y)) \leq \varphi_{\mathcal{E}}((x, k), y) \quad \text{for all} \quad (x, k, y) \in U_1 \times V_1 \times W_1.$$

The next result gives a point-based condition for metric regularity of the epigraphical multifunction.

**Theorem 4.3** *Let $X, Y$ be Asplund spaces, and let $F, G : X \rightrightarrows Y$ be closed multifunctions, and let $(\bar{x}, \bar{k}, \bar{y}) \in X \times Y \times Y$ be such that $\bar{y} \in F(\bar{x}) + \bar{k}, \bar{k} \in G(\bar{x})$. Suppose that*

*(i) $F$ or $G$ is PSNC at $(\bar{x}, \bar{y} - \bar{k})$ and $(\bar{x}, \bar{k})$, respectively;*

*(ii) $D^\star F(\bar{x}, \bar{y} - \bar{k})(0) \cap -D^\star G(\bar{x}, \bar{k})(0) = \{0\}$;*

*(iii) for any $u_n^\star \in \hat{D}^\star F(x_n, y_n - k_n)(y_n^\star + (1/n)B_{Y^\star}), v_n^\star \in \hat{D}^\star G(x_n, k_n)(y_n^\star)$ such that*

$$\|u_n^\star + v_n^\star\| \to 0, y_n^\star \xrightarrow{w^\star} 0 \text{ it follows that } y_n^\star \to 0;$$

*Under the condition that*

$$Ker\big(D^\star F(\bar{x}, \bar{y} - \bar{k}) + D^\star G(\bar{x}, \bar{k})\big) = \{0\}, \tag{43}$$

*the multifunction $\mathcal{E}_{(F,G)}$ is metrically regular around $(\bar{x}, \bar{k}, \bar{y})$.*

*Proof* We prove the result by contradiction. Suppose that $\mathcal{E}_{(F,G)}$ fails to be metrically regular around $(\bar{x}, \bar{k}, \bar{y})$. Then, by Theorem 4.2, there exist sequences

$$(x_n, y_n - k_n) \xrightarrow{F} (\bar{x}, \bar{y} - \bar{k}), (x_n, k_n) \xrightarrow{G} (\bar{x}, \bar{k}), (x_n^\star, u_n^\star, y_n^\star, z_n^\star) \in X^\star \times X^\star \times Y^\star \times Y^\star,$$

with

$$\begin{aligned} &x_n^\star \in \hat{D}^\star F(x_n, y_n - k_n)(y_n^\star + z_n^\star) + u_n^\star, \\ &u_n^\star \in \hat{D}^\star G(x_n, k_n)(y_n^\star), \\ &y_n^\star \in S_{Y^\star}, z_n^\star \in (1/n)B_{Y^\star}, \end{aligned}$$

and

$$x_n^\star \to 0.$$

Then there is $v_n^\star \in \hat{D}^\star F(x_n, y_n - k_n)(y_n^\star + z_n^\star)$ such that $x_n^\star = u_n^\star + v_n^\star$.

Since $Y$ is an Asplund space, we can assume that $y_n^\star \xrightarrow{w^\star} y^\star \in Y^\star$.

We consider the following cases:

Case 1. The sequences $\{u_n^\star\}_{n\in\mathbb{N}}$, $\{v_n^\star\}_{n\in\mathbb{N}}$ are unbounded. We can assume that

$$\|u_n^\star\| \to \infty, \|v_n^\star\| \to \infty,$$

and

$$\frac{u_n^\star}{\|u_n^\star\|} \xrightarrow{w^\star} u^\star, \frac{v_n^\star}{\|v_n^\star\|} \xrightarrow{w^\star} v^\star.$$

Then,

$$y_n^\star/\|u_n^\star\| \to 0 \text{ and } (y_n^\star + z_n^\star)/\|v_n^\star\| \to 0.$$

Consequently,

$$u^\star \in D^\star G(\bar{x}, \bar{k})(0), v^\star \in D^\star F(\bar{x}, \bar{y} - \bar{k})(0).$$

On the other hand,

$$u^\star + v^\star = 0, (\text{since}\|u_n^\star + v_n^\star\| \to 0).$$

It follows that

$$v^\star \in D^\star F(\bar{x}, \bar{y} - \bar{k})(0) \cap -D^\star G(\bar{x}, \bar{k})(0).$$

Therefore, by (ii), we have that $u^\star = v^\star = 0$.

So,

$$\frac{u_n^\star}{\|u_n^\star\|} \to 0, \quad \text{or} \quad \frac{v_n^\star}{\|v_n^\star\|} \to 0, \text{ (by PSNC property of F or G).}$$

This contradicts the fact that $\frac{u_n^\star}{\|u_n^\star\|}$ and $\frac{v_n^\star}{\|v_n^\star\|}$ belong to the unit sphere $S_{Y^\star}$ of $Y^\star$.

Case 2. The sequences $\{u_n^\star\}_{n\in\mathbb{N}}$, $\{v_n^\star\}_{n\in\mathbb{N}}$ are bounded. Assume that $u_n^\star \xrightarrow{w^\star} u^\star$ , $v_n^\star \xrightarrow{w^\star} v^\star$.

It follows that

$$u^\star \in D^\star G(\bar{x}, \bar{k})(y^\star), v^\star \in D^\star F(\bar{x}, \bar{y} - \bar{k})(y^\star).$$

Moreover,

$$u^\star + v^\star = 0.$$

So,

$$0 \in [D^\star G(\bar{x}, \bar{k})(y^\star) + D^\star F(\bar{x}, \bar{y} - \bar{k})(y^\star)] = [D^\star G(\bar{x}, \bar{k}) + D^\star F(\bar{x}, \bar{y} - \bar{k})](y^\star),$$

which means that

$$y^\star \in Ker[D^\star G(\bar{x}, \bar{k}) + D^\star F(\bar{x}, \bar{y} - \bar{k})].$$

By $(\star)$, one has that $y^\star = 0$.

Now, by assumption, one gets $y_n^\star \to 0$ which contradicts to $\|y_n^\star\| = 1$. △

*Remark 4.3* If $X, Y$ are finite dimensional spaces, then conditions (i), (iii) hold true automatically, while condition (ii) holds if $F$ or $G$ is pseudo-Lipschitz at $(\bar{x}, \bar{y} - \bar{k})$ or $(\bar{x}, \bar{k})$, respectively.

## 5 Applications to Variational Systems

In this section, we use the above results to study some properties of variational systems of the form

$$0 \in F(x) + G(x,p), \tag{44}$$

where $X$ is a complete metric space, $Y$ is a Banach space, $P$ is a topological space considered as a parameter space, $F : X \rightrightarrows Y, G : X \times P \rightrightarrows Y$ are given multifunctions. The solution set of (44) is defined by

$$\mathbf{S}_{(F+G)}(p) := \{x \in X : 0 \in F(x) + G(x,p)\}, \tag{45}$$

and we denote

$$\mathbb{S}_{(F+G)}(y,p) := \{x \in X : y \in F(x) + G(x,p)\}.$$

For every $(y,p) \in Y \times P$,

$$\mathbb{S}_{\mathcal{E}_{(F,G)}}(y,p) = \{(x,k) \in X \times Y : y \in F(x) + k, k \in G(x,p)\},$$

and, for every $p \in P$,

$$\mathbf{S}_{\mathcal{E}_{(F,G)}}(p) = \{(x,k) \in X \times Y : 0 \in F(x) + k, k \in G(x,p)\}.$$

We say that the multifunction $\mathbf{S}_{(F+G)}$ is Robinson metrically regular (see [62, 63]) around $(\bar{x},\bar{p})$ with modulus $\tau$, iff there exist neighborhoods $\mathcal{U},\mathcal{V}$ of $\bar{x},\bar{p}$, respectively, such that

$$d(x,\mathbf{S}_{(F+G)}(p)) \le \tau d(0, F(x) + G(x,p)), \text{ for all } (x,p) \in \mathcal{U} \times \mathcal{V}.$$

We also recall that the multifunction $G : X \times P \rightrightarrows Y$ is said to be pseudo-Lipschitz around $(\bar{x},\bar{p},\bar{y})$ with $\bar{y} \in G(\bar{x},\bar{p})$ with respect to $x$, uniformly in $p$ with constant $\kappa > 0$ iff there is a neighborhood $\mathcal{U} \times \mathcal{V} \times \mathcal{W}$ of $(\bar{x},\bar{p},\bar{y})$ such that

$$G(x,p) \cap \mathcal{W} \subset G(u,p) + \kappa d(x,u)\bar{B}_Y \text{ for all } x,u \in \mathcal{U}, \text{and for all } p \in \mathcal{V}.$$

The lower semicontinuous envelope $(x,p,k,y) \mapsto \varphi_{p,\mathcal{E}}((x,k),y)$ of the distance function $d(y,\mathcal{E}_{(F,G)}((x,p),k))$ is defined by, for each $(x,p,k,y) \in X \times P \times Y \times Y$

$$\varphi_{p,\mathcal{E}}((x,k),y) := \liminf_{(u,v,w)\to(x,k,y)} d(w,\mathcal{E}_{(F,G)}((u,p),v))$$

$$= \begin{cases} \liminf\limits_{(u,v)\to(x,k), v\in G(u,p)} d(y, F(u)+k), & \text{if } k \in G(x,p) \\ +\infty, & \text{otherwise.} \end{cases}$$

**Lemma 5.1** *Let $X$ be a complete metric space and $Y$ be a Banach space and let $P$ be a topological space. Suppose that the set-valued mappings $F : X \rightrightarrows Y$, $G : X \times P \rightrightarrows Y$ satisfy the following conditions for some $(\bar{x}, \bar{k}, \bar{p}) \in X \times Y \times P$:*

*(a) $(\bar{x}, \bar{k}) \in \mathbf{S}_{\mathcal{E}_{(F,G)}}(\bar{p})$;*

*(b) the set-valued mapping $p \rightrightarrows G(\bar{x}, p)$ is lower semicontinuous at $\bar{p}$;*

*(c) the set-valued mapping $F$ is a closed multifunction, and for any $p$ near $\bar{p}$, the set-valued mapping $x \rightrightarrows G(x, p)$ is a closed multifunction.*

*Then*

*(i) for ever $p$ near $\bar{p}$, the epigraphical multifunction $\mathcal{E}_{(F,G)}$ has closed graph, and, $\mathcal{E}_{(F,G)}((\bar{x}, \cdot), \bar{k})$ is lower semicontinuous at $\bar{p}$;*

*(ii) the function $p \mapsto \varphi_{p,\mathcal{E}}((\bar{x}, \bar{k}), 0)$ is upper semicontinuous at $\bar{p}$;*

*(iii) for each $(y, p) \in Y \times P$;*

$$\{(x, k) \in X \times Y : \varphi_{p,\mathcal{E}}((x, k), y) = 0\} = \mathbb{S}_{\mathcal{E}_{(F,G)}}(y, p).$$

*Proof* We only note that, if the multifunction $p \rightrightarrows G(\bar{x}, p)$ is lower semicontinuous at $\bar{p}$, then so is the mapping $\mathcal{E}_{(F,G)}((\bar{x}, \cdot), \bar{k})$. △

By using the strong slope of the lower semicontinuous envelope $\varphi_{p,\mathcal{E}}$, one has the following result.

**Theorem 5.1** *Let $X$ be a complete metric space, $Y$ be a Banach space and let $P$ be a topological space. Suppose that the set-valued mappings $F : X \rightrightarrows Y, G : X \times P \rightrightarrows Y$ satisfy conditions $(a), (b), (c)$ from Lemma 5.1 around $(\bar{x}, \bar{k}, \bar{p}) \in X \times Y \times P$. If there exist a neighborhood $\mathcal{T}_1 \times \mathcal{U}_1 \times \mathcal{V}_1 \times \mathcal{W}_1$ of $(\bar{x}, \bar{p}, \bar{k}, 0)$ and reals $m, \gamma > 0$ such that $|\nabla\varphi_{p,\mathcal{E}}((\cdot, \cdot), y)|(x, k) \geq m$ for all $(x, p, k, y) \in \mathcal{T}_1 \times \mathcal{U}_1 \times \mathcal{V}_1 \times \mathcal{W}_1$ with $\varphi_{p,\mathcal{E}}((x, k), y) \in ]0, \gamma[$, then there exists a neighborhood $\mathcal{T} \times \mathcal{U} \times \mathcal{V} \times \mathcal{W}$ of $(\bar{x}, \bar{p}, \bar{k}, 0)$ such that*

$$md((x, k), \mathbb{S}_{\mathcal{E}_{(F,G)}}(y, p)) \leq \varphi_{p,\mathcal{E}}((x, k), y),$$

*for all $(x, p, k, y) \in \mathcal{T} \times \mathcal{U} \times \mathcal{V} \times \mathcal{W}$.*

*Proof.* Applying Theorem 3.2 and Lemma 5.1 for the mapping $\mathcal{E}_{(F,G)}(\cdot, \cdot)$, one obtains the proof.

**Proposition 5.1** *Let $X$ be a complete metric space and $Y$ be a Banach space and let $P$ be a topological space. Suppose that the set-valued mappings $F : X \rightrightarrows Y, G : X \times P \rightrightarrows Y$ satisfy conditions $(a), (b), (c)$ from Lemma 5.1 around $(\bar{x}, \bar{k}, \bar{p}) \in X \times Y \times P$. If there exist a neighborhood $\mathcal{T} \times \mathcal{U} \times \mathcal{V} \times \mathcal{W} \subset X \times P \times Y \times Y$ of $(\bar{x}, \bar{p}, \bar{k}, 0)$ and $m > 0$ such that*

$$md((x, k), \mathbb{S}_{\mathcal{E}_{(F,G)}}(y, p)) \leq \varphi_{p,\mathcal{E}}((x, k), y) \quad \textit{for all} \quad (x, p, k, y) \in \mathcal{T} \times \mathcal{U} \times \mathcal{V} \times \mathcal{W}$$

*then there exists $\theta > 0$ such that*

$$md(x, \mathbb{S}_{(F+G)}(y,p)) \le d(y, F(x) + G(x,p) \cap B(\bar{k}, \theta)) \quad \text{for all} \quad (x,p,y) \in \mathcal{T} \times \mathcal{U} \times \mathcal{W}.$$

*Therefore,*

$$md(x, \mathbf{S}_{(F+G)}(p)) \le d(0, F(x) + G(x,p) \cap B(\bar{k}, \theta)) \quad \text{for all} \quad (x,p) \in \mathcal{T} \times \mathcal{U}.$$

*Proof* By the hypothesis, there exist a neighborhood $\mathcal{T} \times \mathcal{U} \times \mathcal{V} \times \mathcal{W} \subset X \times P \times Y \times Y$ of $(\bar{x}, \bar{p}, \bar{k}, 0)$ and $m > 0$ such that, for every $(x,p,k,y) \in \mathcal{T} \times \mathcal{U} \times \mathcal{V} \times \mathcal{W}$, it holds

$$md((x,k), \mathbb{S}_{\mathcal{E}_{(F,G)}}(y,p)) \le \varphi_{p,\mathcal{E}}((x,k), y).$$

Here, we can assume $\mathcal{V} = B(\bar{k}, \theta)$, with certain positive $\theta$. Then, for every small $\varepsilon > 0$ and for every $(x,p,k,y) \in \mathcal{T} \times \mathcal{U} \times [B(\bar{k}, \theta) \cap G(x,p)] \times \mathcal{W}$, there is $(u,z) \in \mathbb{S}_{\mathcal{E}_{(F,G)}}(y,p)$, i.e., $y \in F(u) + z, z \in G(u,p)$ such that

$$md(u,x) \le m \max\{d(u,x), \|z - k\|\} < (1+\varepsilon) d(y, F(x) + k).$$

Noting that $u \in (F+G)^{-1}(y)$, we obtain that

$$md(x, (F+G)^{-1}(y)) < (1+\varepsilon) d(y, F(x) + k).$$

Thus,

$$md(x, (F+G)^{-1}(y)) \le (1+\varepsilon) d(y, F(x) + G(x,p) \cap B(\bar{k}, \theta)),$$

or,

$$md(x, \mathbb{S}_{(F+G)}(y,p)) \le (1+\varepsilon) d(y, F(x) + G(x,p) \cap B(\bar{k}, \theta)).$$

Since this inequality does not depend on arbitrarily small $\varepsilon > 0$, we obtain that

$$md(x, \mathbb{S}_{(F+G)}(y,p)) \le d(y, F(x) + G(x,p) \cap B(\bar{k}, \theta))$$

for all $(x,p,y) \in \mathcal{T} \times \mathcal{U} \times \mathcal{W}$.

Taking $\bar{y} = 0$ and $y = \bar{y}$, we obtain the second conclusion of the Theorem. The proof is complete. △

In the sequel, we use for the parametrized case the concept of locally sum-stability, which was considered in the previous section.

**Definition 5.1** Let $F : X \rightrightarrows Y, G : X \times P \rightrightarrows Y$ be two multifunctions and $(\bar{x}, \bar{p}, \bar{y}, \bar{z}) \in X \times P \times Y \times Y$ be such that $\bar{y} \in F(\bar{x}), \bar{z} \in G(\bar{x}, \bar{p})$. We say that the pair $(F,G)$ is locally sum-stable around $(\bar{x}, \bar{p}, \bar{y}, \bar{z})$ iff, for every $\varepsilon > 0$, there exists $\delta > 0$ and a neighborhood $W$ of $\bar{p}$ such that, for every $(x,p) \in B(\bar{x}, \delta) \times W$ and every $w \in (F+G)(x) \cap B(\bar{y} + \bar{z}, \delta)$, there are $y \in F(x) \cap B(\bar{y}, \varepsilon)$ and $z \in G(x) \cap B(\bar{z}, \varepsilon)$ such that $w = y + z$.

A following simple case which ensures the locally sum-stability of the pair $(F, G)$, is analogous to Proposition 3.2.

**Proposition 5.2** *Let $F : X \rightrightarrows Y, G : X \times P \rightrightarrows Y$ be two multifunctions and $(\bar{x}, \bar{p}, \bar{y}, \bar{z}) \in X \times P \times Y \times Y$ such that $\bar{y} \in F(\bar{x}), \bar{z} \in G(\bar{x}, \bar{p})$. If $G(\bar{x}, \bar{p}) = \{\bar{z}\}$ and $G$ is upper semicontinuous at $(\bar{x}, \bar{p})$, then the pair $(F, G)$ is locally sum-stable around $(\bar{x}, \bar{p}, \bar{y}, \bar{z})$.*

**Proposition 5.3** *Let $X$ be a complete metric space, $Y$ be a Banach space and let $P$ be a topological pace. Suppose that the set-valued mappings $F : X \rightrightarrows Y, G : X \times P \rightrightarrows Y$ satisfy conditions $(a), (b), (c)$ from Lemma 5.1 around $(\bar{x}, \bar{k}, \bar{p}) \in X \times Y \times P$. If there exist a neighborhood $\mathcal{T} \times \mathcal{U}$ of $(\bar{x}, \bar{p})$ and $\theta, \tau > 0$ such that*

$$d(x, \mathbf{S}_{(F+G)}(p)) \le \tau d(0, F(x) + G(x,p) \cap B(\bar{k}, \theta)) \quad \text{for all} \quad (x,p) \in \mathcal{T} \times \mathcal{U}, \tag{46}$$

*and $(F, G)$ is locally sum-stable around $(\bar{x}, \bar{p}, -\bar{k}, \bar{k})$, then $\mathbf{S}_{(\mathbf{F}+\mathbf{G})}$ is Robinson metrically regular around $(\bar{x}, \bar{p})$ with modulus $\tau$.*

*The conclusion remains true if the assumption of local sum stability around $(\bar{x}, \bar{p}, -\bar{k}, \bar{k})$ is replaced by the following one: $G(\bar{x}, \bar{p}) = \{\bar{z}\}$ and $G$ is upper semicontinuous at $(\bar{x}, \bar{p})$.*

*Proof* The proof of this proposition is very similar to that of Proposition 3.3. Here, we sketch the proof. Suppose that (46) holds for every $(x, p) \in \mathcal{T} \times \mathcal{U}$. Here, we can assume that $\mathcal{T} = B(\bar{x}, \delta)$, with some positive $\delta > 0$.

Since $(F, G)$ is locally sum-stable around $(\bar{x}, \bar{p}, -\bar{k}, \bar{k})$, there exists $\delta > 0$ such that, for every $(x, p) \in B(\bar{x}, \delta) \times \mathcal{U}$ and every $w \in (F + G)(x) \cap B(0, \delta)$, there are $y \in F(x) \cap B(-\bar{k}, \theta)$ and $z \in G(x) \cap B(\bar{k}, \theta)$ such that $w = y + z$.

Fix $(x, p) \in B(\bar{x}, \delta) \times \mathcal{U}$. We consider two following cases:

Case 1. $d(0, F(x) + G(x, p)) < \delta/2$. Fix $\gamma > 0$, small enough so that $d(0, F(x) + G(x, p)) + \gamma < \delta/2$, and take $t \in F(x) + G(x, p)$ such that

$$\|t\| < d(0, F(x) + G(x,p)) + \gamma.$$

Hence we have $\|t\| < \delta/2$, i.e., $t \in B(0, \delta/2) \subset B(0, \delta)$. It follows that $t \in [F(x) + G(x, p)] \cap B(0, \delta)$. Therefore, there are $y \in F(x) \cap B(-\bar{k}, \theta)$ and $z \in G(x, p) \cap B(\bar{k}, \theta)$ such that $t = y + z$.

Consequently,

$$t \in F(x) \cap B(-\bar{k}, \theta) + G(x,p) \cap B(\bar{k}, \theta) \subset F(x) + G(x,p) \cap B(\bar{k}, \theta).$$

It follows that

$$d(0, F(x) + G(x,p) \cap B(\bar{k}, \theta)) \le \|t\|.$$

This yields

$$d(0, F(x) + G(x,p) \cap B(\bar{k}, \theta)) < d(0, F(x) + G(x,p)) + \gamma,$$

and therefore, as $\gamma > 0$ is arbitrarily small, we derive that

$$d(0, F(x) + G(x,p) \cap B(\bar{k}, \theta)) \leq d(0, F(x) + G(x,p)).$$

By (46), one derives

$$d(x, \mathbf{S}_{(F+G)}(p)) \leq \tau d(0, F(x) + G(x,p)), \quad \text{for all} \quad (x,p) \in B(\bar{x}, \delta) \times \mathcal{U}.$$

Case 2. $d(0, F(x) + G(x,p)) \geq \delta/2$. According to condition (c), the multifunction $p \rightrightarrows G(\bar{x}, \cdot)$ is lower semicontinuous at $\bar{p}$. It follows that the distance function $d(0, F(\bar{x}) + G(\bar{x}, \cdot))$ is upper semicontinuous at $\bar{p}$, and thus, there exists a neighborhood $W$ of $\bar{p}$ such that

$$d(0, F(\bar{x}) + G(\bar{x}, p) \leq \delta/4, \text{ for all } p \in W.$$

Shrinking $W$ smaller if necessary, we can assume that $W \subset \mathcal{U}$. Choosing $0 < \delta_1 < \min\{\delta, \tau\delta/4\}$. For every $(x,p) \in B(\bar{x}, \delta_1) \times W$, and for every small $\varepsilon > 0$, there exists $u \in \mathbf{S}_{(F+G)}(p)$ such that

$$d(\bar{x}, u) \leq (1 + \varepsilon)\tau d(0, F(\bar{x}) + G(\bar{x}, p)).$$

So,

$$\begin{aligned} d(x,u) \leq d(x, \bar{x}) + d(\bar{x}, u) & \\ & < \delta_1 + \tau(1+\varepsilon) d(0, F(\bar{x}) + G(\bar{x}, p)) \\ & < \tau\delta/4 + \tau(1+\varepsilon)\delta/4 \\ & \leq \tau/2 d(0, F(x) + G(x,p)) \\ & + \tau/2(1+\varepsilon) d(0, F(x) + G(x,p). \end{aligned}$$

Taking the limit as $\varepsilon > 0$ goes to 0, it follows that

$$d(x, \mathbf{S}_{(F+G)}(p)) \leq \tau d(0, F(x) + G(x,p)),$$

establishing the proof. △

The following theorem establishes the Lipschitz property for the solution mapping $\mathbb{S}_{\mathcal{E}_{(F,G)}}$.

**Theorem 5.2** *Let $X$ be a complete metric space, $Y$ be a Banach space, $P$ be a topological space. Suppose that $F : X \rightrightarrows Y$ and $G : X \times P \rightrightarrows Y$ are multifunctions satisfying conditions (a), (b), (c) in Lemma 5.1.*

*If $F$ is metrically regular around $(\bar{x}, -\bar{k})$ with modulus $\tau > 0$ and $G$ is pseudo-Lipschitz around $(\bar{x}, \bar{p}, \bar{k})$ with respect to $x$, uniformly in $p$ with modulus $\lambda > 0$ such that $\tau\lambda < 1$, then $\mathcal{E}_{(F,G)}$ is metrically regular around $(\bar{x}, \bar{p}, \bar{k}, 0)$ with respect to $(x, k)$, uniformly in $p$, with modulus $(\tau^{-1} - \lambda)^{-1}$.*

*Moreover, assume in addition that $P$ be a metric space. If $G$ is pseudo-Lipschitz around $(\bar{x}, \bar{p}, \bar{k})$ with respect to $p$, uniformly in $x$ with modulus $\gamma > 0$, then $\mathbb{S}_{\mathcal{E}_{(F,G)}}$ is pseudo-Lipschitz around $((0, \bar{p}), (\bar{x}, \bar{k}))$ with modulus $L = \gamma + (\gamma + 1)(\tau^{-1} - \lambda)^{-1})$. In particular, $\mathbf{S}_{\mathcal{E}_{(F,G)}}$ is pseudo-Lipschitz around $((0, \bar{p}), (\bar{x}, \bar{k}))$ with modulus $\gamma(1 + (\tau^{-1} - \lambda)^{-1})$.*

*Proof* The first part is the parametrized version of Theorem 3.3. Its proof is completely similar to the one of Theorem 3.3, and is omitted. For the second part, as $\mathcal{E}_{(F,G)}$ is metrically regular around $(\bar{x}, \bar{p}, \bar{k}, 0)$ with respect to $(x, k)$, uniformly in $p$, with modulus $(\tau^{-1} - \lambda)^{-1}$, there exists $\delta_1 > 0$ such that

$$d((x, k), \mathbb{S}_{\mathcal{E}_{(F,G)}}(y, p)) \leq (\tau^{-1} - \lambda)^{-1} \varphi_{p,\mathcal{E}}((x, k), y), \tag{47}$$

for all $(x, p, k, y) \in B((\bar{x}, \bar{p}, \bar{k}, 0), \delta_1)$.

Now, if $G$ is pseudo-Lipschitz around $(\bar{x}, \bar{p}, \bar{k})$ with respect to $p$, uniformly in $x$ with modulus $\gamma > 0$ then there is $\delta_2 > 0$ such that

$$G(x, p) \cap B(\bar{k}, \delta_2) \subset G(x, p') + \gamma d(p, p') \bar{B}_Y, \tag{48}$$

for all $p, p' \in B(\bar{p}, \delta_2)$, for all $x \in B(\bar{x}, \delta_2)$.

Set $\alpha := \min\{\delta_1/(\gamma+1), \delta_2\}$. Fix $(y, p), (y', p') \in B(0, \alpha) \times B(\bar{p}, \alpha)$. Take $(x, k) \in \mathbb{S}_{\mathcal{E}_{(F,G)}}(y, p)) \cap [B(\bar{x}, \alpha) \times B(\bar{k}, \alpha)]$.

Since $(x, k) \in \mathbb{S}_{\mathcal{E}_{(F,G)}}(y, p)) \cap [B(\bar{x}, \alpha) \times B(\bar{k}, \alpha)]$, then

$$y \in F(x) + k, k \in G(x, p) \text{ and } (x, k) \in B(\bar{x}, \alpha) \times B(\bar{k}, \alpha).$$

Along with (48), we can find that $k' \in G(x, p')$ such that

$$\|k - k'\| \leq \gamma d(p, p') < \gamma\alpha,$$

which follows that $k' \in B(\bar{k}, \delta_1)$. Therefore, by (47), one has

$$\begin{aligned} d((x, k'), \mathbb{S}_{\mathcal{E}_{(F,G)}}(y', p')) &\leq (\tau^{-1} - \lambda)^{-1} \varphi_{p',\mathcal{E}}((x, k'), y'), \\ &\leq (\tau^{-1} - \lambda)^{-1} d(y', F(x) + k')), \end{aligned}$$

Hence, by noting that $y \in F(x) + k$, one deduces that

$$\tag{49}$$

$$
\begin{aligned}
d((x,k), \mathbb{S}_{\mathcal{E}_{(F,G)}}(y',p')) &\leq \|k-k'\| + d((x,k'), \mathbb{S}_{\mathcal{E}_{(F,G)}}(y',p')) \\
&\leq \gamma d(p,p') + (\tau^{-1}-\lambda)^{-1} d(y', F(x)+k')), \\
&\leq \gamma d(p,p') + (\tau^{-1}-\lambda)^{-1} (\|y-y'\| + \|k-k'\|) \\
&\leq \gamma(1+(\tau^{-1}-\lambda)^{-1}) d(p,p') + (\tau^{-1}-\lambda)^{-1}\|y-y'\|
\end{aligned}
$$

and so

$$
\begin{aligned}
&\mathbb{S}_{\mathcal{E}_{(F,G)}}(y,p)) \cap [B(\bar{x},\alpha)\times B(\bar{k},\alpha)] \\
&\subseteq \mathbb{S}_{\mathcal{E}_{(F,G)}}(y',p') + Ld((y',p'),(y,p))\bar{B}_X \times \bar{B}_Y,
\end{aligned}
$$

where, $L = \gamma + (\gamma+1)(\tau^{-1}-\lambda)^{-1}$, and by taking $y = y' = 0$ in relation (49), one also derives that $\mathbf{S}_{\mathcal{E}_{(F,G)}}$ is pseudo-Lipschitz around $((0,\bar{p}),(\bar{x},\bar{k}))$ with modulus $\gamma(1+(\tau^{-1}-\lambda)^{-1})$.

The proof is complete. △

If we add the assumption that $(F,G)$ is locally sum-stable, we obtain the Lipschitz property of $\mathbf{S}_{(F+G)}$.

**Theorem 5.3** *Let $X$ be a complete metric space and $Y$ be a Banach space, $P$ be a metric space. Suppose that $F : X \rightrightarrows Y$ and $G : X \times P \rightrightarrows Y$ satisfy conditions (a), (b), (c) in Lemma 5.1. Moreover, assume that*

*(i) $(F,G)$ is locally sum-stable around $(\bar{x},\bar{p},-\bar{k},\bar{k})$;*

*(ii) $F$ is metrically regular around $(\bar{x},-\bar{k})$ with modulus $\tau > 0$;*

*(iii) $G$ is pseudo-Lipschitz around $(\bar{x},\bar{p},\bar{k})$ with respect to $x$, uniformly in $p$ with modulus $\lambda > 0$ such that $\tau\lambda < 1$;*

*(iv) $G$ is pseudo-Lipschitz around $(\bar{x},\bar{p},\bar{k})$ with respect to $p$, uniformly in $x$ with modulus $\gamma > 0$. Then $\mathbf{S}_{(F+G)}$ is Robinson metrically regular around $(\bar{x},\bar{p})$ with modulus $(\tau^{-1}-\lambda)^{-1}$. Moreover, $\mathbf{S}_{(F+G)}$ is pseudo-Lipschitz around $(\bar{x},\bar{p})$ with constant $\gamma(\tau^{-1}-\lambda)^{-1}$.*

*Proof* Applying Proposition 5.2, Proposition 23 and Proposition 20, respectively, we obtain that $S_{(F+G)}$ is Robinson metrically regular around $(\bar{x},\bar{p})$ with modulus $(\tau^{-1}-\lambda)^{-1}$. Thus, there exists $\delta_1 > 0$ such that

$$d(x, \mathbf{S}_{(F+G)}(p)) \leq (\tau^{-1}-\lambda)^{-1} d(0, F(x)+G(x,p)), \text{ for all } (x,p) \in B((\bar{x},\bar{p}),\delta_1).$$

On the other hand, since $G$ is pseudo-Lipschitz around $(\bar{x},\bar{p},\bar{k})$ with respect to $p$, uniformly in $x$ with modulus $\gamma > 0$, we can find $\delta_2 > 0$ such that

$$G(x,p) \cap B(\bar{k},\delta_2) \subset G(x,p') + \gamma d(p,p')\bar{B}_Y,$$

for all $p, p' \in B(\bar{p}, \delta_2)$, for all $x \in B(\bar{x}, \delta_2)$. Moreover, since the pair $(F, G)$ is locally sum-stable around $(\bar{x}, \bar{p}, -\bar{k}, \bar{k})$, there is $\delta_3 > 0$ such that, for every $(x, p) \in B(\bar{x}, \delta_3) \times B(\bar{p}, \delta_3)$ and every $w \in [F(x) + G(x, p)] \cap B(0, \delta_3)$, there are $y \in F(x) \cap B(-\bar{k}, \delta_2), z \in G(x, p) \cap B(\bar{k}, \delta_2)$ such that $w = y + z$. Set $\alpha := \min\{\delta_1, \delta_2, \delta_3\}$. Take $p, p' \in B(\bar{p}, \alpha)$, and $x \in \mathbf{S}_{(F+G)}(p) \cap B(\bar{x}, \alpha)$, i.e., $0 \in F(x) + G(x, p)$ and $x \in B(\bar{x}, \alpha)$.

Moreover, we observe that for every $w \in [F(x) + G(x, p)] \cap B(0, \alpha)$,

$$w \in F(x) \cap B(-\bar{k}, \delta_2)) + G(x, p) \cap B(\bar{k}, \delta_2) \subseteq F(x) + G(x, p') + \gamma d(p, p')\bar{B}_Y.$$

Thus,

$$[F(x) + G(x, p)] \cap B(0, \alpha) \subseteq F(x) + G(x, p') + \gamma d(p, p')\bar{B}_Y.$$

Since $0 \in F(x) + G(x, p)$, and also $0 \in [F(x) + G(x, p)] \cap B(0, \alpha)$, thus

$$0 \in F(x) + G(x, p') + \gamma d(p, p')\bar{B}_Y.$$

It follows that there is $w \in F(x) + G(x, p')$ such that $\|w\| \leq \gamma d(p, p')$. Therefore,

$$d(x, \mathbf{S}_{(F+G)}(p')) \leq (\tau^{-1} - \lambda)^{-1} d(0, F(x) + G(x, p')) \leq (\tau^{-1} - \lambda)^{-1}\|w\| \leq \gamma(\tau^{-1} - \lambda)^{-1} d(p, p').$$

So,

$$\mathbf{S}_{(F+G)}(p) \cap B(\bar{x}, \alpha) \subseteq \mathbf{S}_{(F+G)}(p') + \gamma(\tau^{-1} - \lambda)^{-1} d(p, p')\bar{B}_X,$$

establishing the proof.

## 6 Concluding Remarks

We conclude the paper with some comments and perspectives on metric regularity/pseudo-Lipschitzness of set-valued mappings and on the study of the associated variational systems. It is not possible to obtain effective results on the Lipschitzness of the sum when the both multifunctions $F$ and $G$ depend on the parameter $p$ (see [54], and [15]). Similarly to [15], we also used variational techniques to obtain the desired variational properties of the sum or to the correspondent variational systems; however, in this article, we used the theory of error bound systematically to study metric regularity of a type of epigraphical multifunction associated to two given set-valued mappings. On one hand, this approach, avoids the closedness of the sum mapping $F+G$, on the other hand, it provides a way to derive variational properties of the system associated to the epigraphical mapping without using the sum-stable property (Theorem 5.2). This method, allows to study more general kinds of multifunctions, such as composition of two set-valued mappings, as well as variational systems associated to them.

Moreover, we also note that if a set-valued mapping $F : X \rightrightarrows Y$ is pseudo-Lipschitz around $(\bar{x}, \bar{y}) \in \operatorname{gph} F$, then it is lower semicontinuous at $\bar{x}$. So, in any results above, if we impose the assumption of pseudo-Lipschitzness to $F$, then the assumption of lower semicontinuity is automatically satisfied.

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